D=6, N=1 → D=4, N=2 NO SCALE SUPERGRAVITY THEORIES + JORDAN ALGEBRAS

"30 YEARS OF SUGRA IN PARIS"

OCT. 17, 2006

M.G.

N=2 MULTIPLETS IN D=5

Graviton	Gravitini	Vector	$s=\frac{1}{2}$	Scalar
$(g_{\mu\nu}$	ψ_μ^i	$A_\mu)$ $\Leftarrow$ graviton multiplet		
Vector Multiplet $\Rightarrow$		$(A_\mu$	λ^i	$\phi)$
Hypermultiplet $\Rightarrow$			$2(\zeta$	$q^i)$

$i=1,2$ $SU(2)_R$ indices

N=2 MULTIPLETS IN D=4

$$(g_{\mu\nu} \quad \psi_\mu^i \quad A_\mu)$$

$$(A_\mu \quad \lambda^i \quad z, \bar{z})$$

$$2(\zeta \quad q^i)$$

N=2 MULTIPLETS IN D=6:

$$(g_{\mu\nu} \quad \psi_\mu^i(L) \quad B^-_{\mu\nu})$$

$$(A_\mu \quad \lambda^i(L))$$

Tensor Multiplet $\Rightarrow$ $(B^+_{\mu\nu} \quad \lambda^i(R) \quad \phi)$

$$2(\zeta \quad q^i)$$

L(R) refer to left-handed (right-handed) spinors of $SO(5,1)$.

$B^+_{\mu\nu}$ $\Rightarrow$ self-dual tensor $\Rightarrow$ $F^+_{\mu\nu\rho} = \partial_{[\mu} B^+_{\nu\rho]} = *F^+_{\mu\nu\rho}$

$B^-_{\mu\nu}$ $\Rightarrow$ anti-self dual tensor $\Rightarrow$ $F^-_{\mu\nu\rho} = - *F^-_{\mu\nu\rho}$

$$(*F)_{\mu\nu\rho} \equiv \frac{1}{6} \epsilon_{\mu\nu\rho\lambda\sigma\epsilon} F^{\lambda\sigma\epsilon}$$

N=2, D=5 MAXWELL-EINSTEIN SUPERGRAVITY

5d, N=2 SUGRA COUPLED TO n VECTOR MULTIPLETS:

MG, SIERRA & TOWNSEND (1983)

$$\left(e_\mu^m + \psi_\mu^i + \underbrace{A_\mu^0 \right) \oplus \left(A_\mu^a}_{A_\mu^I} + \lambda^{ai} + \phi^a \right)$$

$$e^{-1}\mathcal{L}_{\text{bosonic}} = -\frac{1}{2}R - \frac{1}{4}\overset{\circ}{a}_{IJ} F^I_{\mu\nu} F^{J\mu\nu} - \frac{1}{2} g_{xy}\, \partial_\mu \varphi^x \partial^\mu \varphi^y$$
$$+ \frac{e^{-1}}{6\sqrt{6}} C_{IJK} F^I_{\mu\nu} F^J_{\lambda\rho} A^K_\sigma \epsilon^{\mu\nu\lambda\rho\sigma}$$

$a, b = 1, \dots n$

$I, J, \dots = 0, 1, \dots, n$

$x, y, \dots = 1, \dots, n$

CONSTANT SYMMETRIC TENSOR C_{IJK}

REMARKABLE FACT: N=2 Maxwell-Einstein supergravity (MESGT) IS UNIQUELY DETERMINED BY C_{IJK}

SCALAR MANIFOLD $\mathcal{M}$ IS AN HYPERSURFACE IN AN $(n+1)$ DIMENSIONAL AMBIENT SPACE $\mathcal{C}$ WITH COORDINATES h^I AND METRIC a_{IJ}:

$$a_{IJ} = -\frac{1}{3}\partial_I \partial_J \ln \mathcal{V}(h)$$

$$\mathcal{V}(h) = C_{IJK} h^I h^J h^K$$

$\mathcal{M}$ IS THE $\mathcal{V}(h) = 1$ HYPERSURFACE

$$\overset{\circ}{a}_{IJ}(\varphi) = a_{IJ}(h)\Big|_{\mathcal{V}=1}$$

$$g_{xy} = \overset{\circ}{a}_{IJ} h^I_x h^J_y \qquad h^I_x \equiv \frac{\partial h^I}{\partial \varphi^x}$$

GLOBAL SYMMETRIES $\equiv$ SYMMETRIES OF C_{IJK}

ELEVEN DIMENSIONAL SUGRA COMPACTIFIED OVER A CALABI-YAU MANIFOLD YIELDS A 5d, N=2 MESGT COUPLED TO N=2 HYPERMULTIPLETS

C_{IJK} = TOPOLOGICAL INTERSECTION NUMBERS OF CY

$h_{1,1} = n_V - 1 \qquad h_{2,1} = n_H - 1$

RIEMANN TENSOR OF SCALAR MANIFOLD $\mathcal{M}$ OF N=2 MESGT

$$R_{xyzu} = \frac{4}{3}\left\{ g_{x[u}\, g_{z]y} + T_{x[u}{}^{w}\, T_{z]yw} \right\}$$

$$T_{xyz} = C_{IJK}\, h^I_x\, h^J_y\, h^K_z$$

IF T_{xyz} IS COVARIANTLY CONSTANT THEN $\mathcal{M}$ IS A SYMMETRIC SPACE

$$T_{xyz;u} = 0 \implies R_{xyzu;w} = 0$$

$T_{xyz;u} = 0$ $\oplus$ N=2 SUSY $\oplus$ POSITIVITY OF K.E IMPLY THAT $\mathcal{V}(h) = C_{IJK} h^I h^J h^K$ CAN BE IDENTIFIED WITH THE NORM FORM OF A EUCLIDEAN JORDAN ALGEBRA J OF DEGREE 3 !

$$\mathcal{M} = \frac{Str_0(J)}{Aut(J)}$$

$Str_0(J)$ = INVARIANCE GROUP OF THE NORM OF J ≡ REDUCED STRUCTURE GROUP OF J

$Aut(J)$ = AUTOMORPHISM GROUP OF J

JORDAN ALGEBRAS

$$X \circ Y = Y \circ X \qquad\qquad X, Y \in J$$

$$X \circ (Y \circ X^2) = (X \circ Y) \circ X^2$$

EUCLIDEAN (FORMALLY REAL) JORDAN ALGEBRAS HAVE THE PROPERTY

$$X^2 + Y^2 = 0 \quad \Rightarrow \quad X = 0 \ \text{AND} \ Y = 0$$

$n \times n$ HERMITIAN MATRICES OVER $A = \mathbb{R}, \mathbb{C}, \mathbb{H}$ FORM A JORDAN ALGEBRA J_n^A WITH THE PRODUCT

$$X \circ Y \equiv \frac{1}{2}(XY + YX)$$

WHICH PRESERVES HERMITICITY.

COMPLETE LIST OF SIMPLE EUCLIDEAN JORDAN ALGEBRAS (FINITE DIMENSIONAL):

$$J_n^{\mathbb{R}}, \quad J_n^{\mathbb{C}}, \quad J_n^{\mathbb{H}}$$

$J_3^{\mathbb{O}}$ = 3x3 Hermitian over real octonions

$\Gamma(D)$ ≡ DIRAC GAMMA MATRICES IN D-DIM'NAL EUCLIDEAN SPACE

$J_3^{\mathbb{O}}$ HAS NO REALIZATION IN TERMS OF ASSOCIATIVE MATRICES WITH THE ABOVE PRODUCT !

$\Leftrightarrow$ EXCEPTIONAL JORDAN ALGEBRA

$\exists$ FOUR SIMPLE JORDAN ALGEBRAS OF DEGREE 3

$$J_3^{\mathbb{R}}, \quad J_3^{\mathbb{C}}, \quad J_3^{\mathbb{H}}, \quad J_3^{\mathbb{O}}$$

AND AN INFINITE FAMILY OF NON-SIMPLE JORDAN ALGEBRAS OF DEGREE 3

$$J = \mathbb{R} \oplus \Gamma(D)$$

QUADRATIC MAP : $X \longrightarrow X^{\#}$

$$(X^{\#})_I = C_{IJK} X^J X^K$$

$$T_{xyz;u} = 0 \implies (X^{\#})^{\#} = \mathcal{V}(X)\, X$$ ADJOINT IDENTITY

$$\mathcal{V}(X) = C_{IJK} X^I X^J X^K$$ cubic norm of a degree 3 Jordan Algebra

SCALAR MANIFOLD of N=2 MESGT:

$$\mathcal{M} = \frac{Str_0(J)}{Aut(J)}$$

Dim J = n+1 = # of vector fields

Dim $\mathcal{M}$ = n = # of scalar fields

GENERIC (REDUCIBLE) JORDAN FAMILY $J = \mathbb{R} \oplus \Gamma(Q)$

$$\mathcal{V} = \alpha Q$$

$\alpha \in \mathbb{R}$, Q is a quadratic form of signature $(+,-,-,\cdots)$

$$\mathcal{M} = \frac{SO(n-1,1) \times SO(1,1)}{SO(n-1)}$$

FOR N=2 MESGT'S DEFINED BY SIMPLE JORDAN ALGEBRAS OF DEGREE 3 :

$$\mathcal{M}(J_3^{\mathbb{R}}) = \frac{SL(3,\mathbb{R})}{SO(3)}$$ (5+1) vectors

$$\mathcal{M}(J_3^{\mathbb{C}}) = \frac{SL(3,\mathbb{C})}{SU(3)}$$ (8+1) vectors

$$\mathcal{M}(J_3^{\mathbb{H}}) = \frac{SU^*(6)}{USp(6)}$$ 14+1 vectors

$$\mathcal{M}(J_3^{\mathbb{O}}) = \frac{E_{6(-26)}}{F_4}$$ 26+1 vectors

NORM FORM $\simeq$ 'DETERMINANT' OF $J_3^{\mathbb{A}}$

NON-JORDAN FAMILY OF $N=2$, $D=5$

MESGT's WITH TARGET SPACES

$$\mathcal{M} = \frac{SO(n,1)}{SO(n)}$$

GST 1986

ONLY THE PARABOLIC SUBGROUP

$$[SO(n-1) \times SO(1,1)] \circledS T_{n-1} \subset SO(n,1)$$

IS A SYMMETRY OF THE LAGRANGIAN

(de Wit & van Proeyen 1992)

CLASSIFICATION OF SCALAR MANIFOLDS $\mathcal{M}$ OF $D=5$, $N=2$ MESGT's THAT ARE HOMOGENEOUS SPACES

dW & vP 1992

HOMOGENEOUS, BUT NOT SYMMETRIC, SCALAR MANIFOLDS ARE OF THE FORM:

$$\mathcal{M} = \frac{G}{H}$$

G = Parabolic Group

H = Maximal Compact subgroup

Lie algebra $g = g^{0} \oplus g^{+1}$

N=2, 5d MESGT'S DEFINED BY SIMPLE JORDAN ALGEBRAS OF DEGREE 3 ARE UNIFIED THEORIES

$\xleftrightarrow{\text{SUSY}}$

$$e_\mu^m \quad \psi_\mu^i \quad \begin{pmatrix} A_\mu^0 \\ A_\mu^a \end{pmatrix} \Big\updownarrow G \quad \lambda^{ia} \quad \phi^a$$

$\xleftrightarrow{\text{SUSY}}$

$$\mathcal{M} = G/H$$

UNIFIED MESGT $\Longleftrightarrow$ C_{IJK} ≡ INVARIANT TENSOR OF A SIMPLE GROUP G

THEY ARE THE ONLY UNIFIED MESGT'S IN $d=5$ WHOSE SCALAR MANIFOLDS ARE SYMMETRIC SPACES !

GST 1983

FURTHERMORE THERE EXIST 3 INFINITE FAMILIES OF UNIFIED MESGT'S IN $d=5$ WHOSE SCALAR MANIFOLDS ARE NOT HOMOGENEOUS SPACES (plus a sporadic one)

M.G + Zagermann 2003

C_{IJK} ≡ STRUCTURE CONSTANTS OF NONCOMPACT (LORENTZIAN) JORDAN ALGEBRAS OF DEGREE > 4

$$J^{\mathbb{R}}_{(1,N)_0}, \quad J^{\mathbb{C}}_{(1,N)_0}, \quad J^{\mathbb{H}}_{(1,N)_0}, \quad N>3$$

(plus $J^{\mathbb{O}}_{(1,2)_0}$)

GENERATED BY MATRICES OVER $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ THAT ARE HERMITIAN WITH RESPECT TO A LORENTZIAN METRIC

$$(X\eta)^\dagger = X\eta \qquad \eta = \mathrm{Diag}(+,-,-,\cdots)$$

UNIFIED $D=5$, $N=2$ MESGT DEFINED BY LORENTZIAN JORDAN ALGEBRAS $(N>3)$

J	G	# of Vector Fields
$J^{\mathbb{R}}_{(1,N)}$	$SO(N,1)$	$\frac{1}{2}N(N+3)$
$J^{\mathbb{C}}_{(1,N)}$	$SU(N,1)$	$N(N+2)$
$J^{\mathbb{H}}_{(1,N)}$	$USp(2N,2)$	$N(2N+3)$
$J^{\mathbb{O}}_{(1,2)}$	$F_{4(-20)}$	26

NOTE THE SPECIAL EQUIVALENCES OF THEORIES DEFINED BY $J^{\mathbb{A}}_{(1,3)}$ AND EUCLIDEAN JORDAN ALGEBRAS $J_3^{\tilde{\mathbb{A}}}$

$$J^{\mathbb{R}}_{(1,3)} \approx J_3^{\mathbb{C}}$$

$$J^{\mathbb{C}}_{(1,3)} \approx J_3^{\mathbb{H}}$$

$$J^{\mathbb{H}}_{(1,3)} \approx J_3^{\mathbb{O}}$$

GAUGING SYMMETRIES OF $D=5$, $N=2$ MESGT

SYMMETRY GROUP OF THE LAGRANGIAN

$$G \times SU(2)_R$$

G = INVARIANCE GROUP OF C_{IJK}

GAUGED MESGT $\Rightarrow$ GAUGE GROUP $\subset SU(2)_R$

$U(1)_R$ GAUGED MESGT'S IN $D=5$

GST (1984)

$$A_\mu = V_I A^I_\mu \iff U(1)_R \text{ Gauge Field}$$

$$\nabla_\mu \lambda^i \longrightarrow (D_\mu \lambda)^i \equiv \nabla_\mu \lambda^i + g_R A_\mu \lambda_j \delta^{ij}$$

$$\nabla_\mu \psi^i_\mu \longrightarrow (D_\mu \psi_\nu)^i \equiv \nabla_\mu \psi^i_\nu + g_R A_\mu \delta^{ij} \psi_{\nu j}$$

POTENTIAL $\quad P^{(R)} = -4\, C^{IJK} V_I V_J h_K$

V_I = constant $\qquad h_K = h_K(\varphi_a)$

C_{IJK} = constant $\qquad C^{IJK} = \overset{\circ}{a}{}^{I\tilde{I}} \overset{\circ}{a}{}^{J\tilde{J}} \overset{\circ}{a}{}^{K\tilde{K}} C_{\tilde{I}\tilde{J}\tilde{K}}$

FOR MESGTs DEFINED BY JORDAN ALGEBRAS OF DEGREE 3 (EUCLIDEAN!)

$$C^{IJK} = C_{IJK} = \text{INVARIANT TENSOR OF } G$$

DEPENDING ON THE CHOICE OF V_I $U(1)_R$ GAUGING LEADS TO A STABLE SUSY AdS_5 GROUND STATE OR TO A MINKOWSKIAN GROUND STATE WITH A SPONTANEOUSLY BROKEN SUSY.

$$\mathcal{P}^R = -4\, C^{IJK} V_I V_J h_K = -4\, V^{\# I} h_I$$

$$V^{\# I} \equiv C^{IJK} V_J V_K$$

DEPENDING ON THE CHOICE OF V_I, $\mathcal{P}^R$ HAS NO CRITICAL POINTS OR ONE CRITICAL POINT (MAXIMUM).

$N=2$, AdS_5 ground state $\Longleftrightarrow$ $V \in \mathcal{C}(J) \equiv$ Domain of positivity of J

$V^{\#} = 0 \Rightarrow \mathcal{P}^{(R)} = 0$ VANISHING POTENTIAL

$\Rightarrow V = E$ $\quad E^2 = E$, $\mathrm{Tr}\, E = 1$ irreducible idempotent

$\Rightarrow$ Minkowskian Ground State with spontaneously broken supersymmetry.

$\Rightarrow$ NO-SCALE SUPERGRAVITY WITH ZERO COSM. CONSTANT AND SPONTAN. BROKEN SUSY.

SCALE OF SUSY BREAKING IS NOT FIXED IN THE CLASSICAL THEORY.

QUANTUM CORRECTIONS MAY MODIFY THE FLAT POTENTIAL AND DETERMINE THE SUSY BREAKING SCALE DYNAMICALLY (refer to talks by Ellis + Zwirner, ..)

IN THE CORRESPONDING CLASSICAL GROUND STATES OF $U(1)_R$ GAUGED MESGTs THE GLOBAL SYMMETRY GROUP G GETS BROKEN TO THE SUBGROUP H OF G THAT LEAVES V INVARIANT.

H = SYMMETRY OF THE GROUND STATE

V = IDENTITY ELEMENT OF JORDAN ALG. J

$\Rightarrow$ N=2 AdS_5 GROUND STATE

J	G	H
$J_3^{\mathbb{R}}$	$SL(3,\mathbb{R})$	$SO(3)$
$J_3^{\mathbb{C}}$	$SL(3,\mathbb{C})$	$SU(3)$
$J_3^{\mathbb{H}}$	$SU^*(6)$	$USp(6)$
$J_3^{\mathbb{O}}$	$E_{6(-26)}$	F_4
$\mathbb{R}+\Gamma$	$\dfrac{SO(n-1)\times SO(1,1)}{SO(n-1)}$	$SO(n-1)$

V = E = AN IRR. IDEMPOTENT OF J

$\Rightarrow$ VANISHING POTENTIAL $P^{(R)}=0$

J	G	H
$J_3^{\mathbb{R}}$	$SL(3,\mathbb{R})$	$SL(2,\mathbb{R})$ Ⓢ $\mathbb{R}^2$
$J_3^{\mathbb{C}}$	$SL(3,\mathbb{C})$	$SL(2,\mathbb{C})\times U(1)$ Ⓢ $\mathbb{C}^2$
$J_3^{\mathbb{H}}$	$SU^*(6)$	$SO(5,1)\times SU(2)$ Ⓢ $\mathbb{H}^2$
$J_3^{\mathbb{O}}$	$E_{6(-26)}$	$SO(9,1)$ Ⓢ $\mathbb{O}^2$
$\mathbb{R}+\Gamma$	$SO(n-1,1)\times SO(1,1)$	$SO(n-1,1)$
$\mathbb{R}+\Gamma$	$SO(n-1,1)\times SO(1,1)$	$\widetilde{SO}(1,1)\times[SO(n-2)$ Ⓢ $T_{n-2}]$

H IS A PARABOLIC SUBGROUP OF G

N=2, D=5, YANG-MILLS / EINSTEIN SUGRAS

(GST 84)

GAUGE A SUBGROUP $K \subset G$ SUCH THAT THE VECTOR FIELDS DECOMPOSE AS THE ADJOINT PLUS SINGLETS OF K

NO POTENTIAL IS INTRODUCED IN D=5

$$\Delta \mathcal{L} = -\frac{i}{2} g \, \bar{\lambda}^{ia} \lambda_i^b K_{I[a} h^I_{b]}$$ ← additional term in Lagrangian

UNIFIED YANG-MILLS / EINSTEIN SUGRAS:

ALL THE VECTOR FIELDS INCLUDIND THE GRAVIPHOTON TRANSFORM IN THE ADJOINT OF K WHICH IS SIMPLE!

UNIQUE UNIFIED YMESGT WHOSE SCALAR MANIFOLD $\mathcal{M}_5$ IS A SYMMETRIC SPACE

$J_3^{\mathbb{H}}$ THEORY WITH $K = SO^*(6) = SU(3,1) \subset SU^*(6)$

OF THE 3 INFINITE FAMILIES OF UNIFIED MESGT's DEFINED BY LORENTZIAN JORDAN ALGEBRAS $J^{\mathbb{A}}_{(1,N)}$ $(\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H})$, THE FAMILY $J^{\mathbb{C}}_{(1,N)}$ CAN BE GAUGED SO AS TO OBTAIN AN INFINITE FAMILY OF UNIFIED YMESGTs WITH GAUGE GROUP $SU(N,1)$

FOR $N > 3$ SCALAR MANIFOLDS ARE NOT HOMOGENEOUS!

M.G & Zagermann (2003)

IN THE MINKOWSKIAN GROUND STATES OF THESE THEORIES THE NON-COMPACT GAUGE GROUPS GET BROKEN DOWN TO THEIR MAXIMAL COMPACT SUBGROUPS!

N=2, D=5 YMESGT'S WITH TENSORS

M.G + Zagermann (2000

If under the subgroup $K \subset G$ there are non-singlet vectors (other than the adjoint of K) then they must be dualized to tensors

$$A_\mu^{\tilde{I}} = (\underbrace{A_\mu^I}_{\text{adjoint + singlets of } K} \oplus \underbrace{A_\mu^M}_{\text{non-singlets of } K})$$

$$A_\mu^M \longrightarrow B_{\mu\nu}^M$$

$B_{\mu\nu}^M$ satisfy 5 dimensional self-duality (Pilch, Townsend, P.v.N.)

$\Rightarrow$ even # of $B_{\mu\nu}^M$ transforming in a symplectic representation of K

GAUGING $K \subset G$ WITH TENSORS $B_{\mu\nu}^M$ TRANSFORMING IN A SYMPLECTIC REPRESENTATION OF K INTRODUSE A POTENTIAL P^T

$$P^T = 2 W^{\tilde{a}} W^{\tilde{a}} \geqslant 0$$

$$W^{\tilde{a}} = \frac{\sqrt{6}}{4} h^J K_J^{\tilde{a}}$$

GAUGED YMESGT WITH TENSORS

simultaneously gauge $U(1)_R \subset SU(2)_R$ and $K \subset G$ with non-trivial tensors

$$e^{-1} \mathcal{L}_{Pot.} = - g^2 P^T(\varphi) - g_R^2 P^R(\varphi)$$

$$A_\mu^{(R)} = V_I A_\mu^I$$

$$P^R = -(P_0)^2 + P^{\tilde{a}} P^{\tilde{a}}$$

$$P_0 = 2 h^I V_I \qquad P^{\tilde{a}} = \sqrt{2}\, h^{I\tilde{a}} V_I \qquad f_{JK}^I V_I = 0$$

DIMENSIONAL REDUCTION OF 5d, N=2 MESGT

5d, N=2 MESGT WITH n VECTOR MULTIPLETS $\longleftrightarrow$ VERY SPECIAL REAL GEOMETRY

$\Downarrow$

4d, N=2 MESGT WITH (n+1) VECTOR MULTIPLETS $\longleftrightarrow$ VERY SPECIAL COMPLEX GEOMETRY

$\Downarrow$

3d, N=4 SUGRA COUPLED TO (n+2) N=4 MATTER MULTIPLETS $\longleftrightarrow$ VERY SPECIAL QUATERNIONIC GEOMETRY

GENERIC JORDAN FAMILY OF N=2 MESGT'S

$$J = \mathbb{R} \oplus \Gamma(Q)$$

$$\mathcal{M}_5 = \frac{SO(n-1,1) \times SO(1,1)}{SO(n-1)}$$

$$\mathcal{M}_4 = \frac{SO(n,2)}{SO(n) \times SO(2)} \times \frac{SU(1,1)}{U(1)}$$

$$\mathcal{M}_3 = \frac{SO(n+2,4)}{SO(n+2) \times SO(4)}$$

PURE N=2, d=5 SUGRA UNDER DIMENSIONAL REDUCTION YIELDS N=2, d=4 SUGRA COUPLED TO ONE VECTOR MULTIPLET WITH

$$\mathcal{M}_4 = \frac{SU(1,1)_G}{U(1)} \Rightarrow \left(F^I_{\mu\nu} \oplus \tilde{F}^J_{\mu\nu} \right) \sim s = \frac{3}{2} \text{ of } SU(1,1)_G$$

FURTHER REDUCTION TO d=3 YIELDS

$$\mathcal{M}_3 = G_{2(2)} / SU(2) \times SU(2)$$

SYMMETRY GROUPS OF $N=2$ MESGT's DEFINED BY SIMPLE (FR) JORDAN ALGEBRAS OF DEGREE 3 :

U-Duality	$J_3^{\mathbb{R}}$	$J_3^{\mathbb{C}}$	$J_3^{\mathbb{H}}$	$J_3^{\mathbb{O}}$
K_5	$SO(3)$	$SU(3)$	$USp(6)$	F_4
U_5	$SL(3,\mathbb{R})$	$SL(3,\mathbb{C})$	$SU^*(6)$	$E_{6(-26)}$
U_4	$Sp(6,\mathbb{R})$	$SU(3,3)$	$SO^*(12)$	$E_{7(-25)}$
U_3	$F_{4(4)}$	$E_{6(2)}$	$E_{7(-5)}$	$E_{8(-24)}$

THE "MAGIC SQUARE" OF FREUDENTHAL, ROZENFELD AND TITS $\Rightarrow$ Magical Supergravity theories

	EXCEPTIONAL SUGRA $J_3^{\mathbb{O}}$	MAXIMAL SUGRA	COMMON SECTOR $= J_3^{\mathbb{H}}$ THEORY
$\mathcal{M}_5 =$	$E_{6(-26)}/F_4$	$E_{6(6)}/USp(8)$	$SU^*(6)/USp(6)$
$\mathcal{M}_4 =$	$E_{7(-25)}/E_6 \times U(1)$	$E_{7(7)}/SU(8)$	$SO^*(12)/U(6)$
$\mathcal{M}_3 =$	$E_{8(-24)}/E_7 \times SU(2)$	$E_{8(8)}/SO(16)$	$E_{7(-5)}/SO(12) \times SU(2)$

DISCRETE SUBGROUPS $E_{7(7)}(\mathbb{Z})$ and $E_{6(6)}(\mathbb{Z})$

$\Rightarrow$ SYMMETRY GROUPS OF NON-PERTURBATIVE SPECTRA OF M-THEORY TOROIDALLY COMPACTIFIED

T-DUALITY $\times$ S-DUALITY $\subset$ U-DUALITY

$SO(6,6) \times SL(2,\mathbb{R}) \subset E_{7(7)}$ Hull & Townsend

$SO(5,5) \times SO(1,1) \subset E_{6(6)}$

ANALOGOUS DECOMPOSITION FOR THE EXCEPTIONAL SUGRA

$SO(10,2) \times SL(2,\mathbb{R}) \subset E_{7(-25)}$

$SO(9,1) \times SO(1,1) \subset E_{6(-26)}$

$N=2$ MESGT in $D=4$

GENERAL CONSTRUCTION USING SUPERCONFORMAL TENSOR CALCULUS   de Wit + van Proeyen et. al. 1983/84

CRUCIAL DIFFERENCES w.r.t $D=5$, MESGT:

- U-DUALITY GROUP G IS, IN GENERAL, ON-SHELL SYMMETRY.
- GAUGING A SUBGROUP H OF G THAT IS A SYMMETRY OF THE LAGRANGIAN INTRODUCES A POSITIVE SEMI-DEFINITE POTENTIAL (recall that the potential vanishes in YMESGTs in $D=5$!)
- SUBGROUP H DEPENDS ON THE SYMPLECTIC SECTION CHOSEN!

STUDY OF GAUGED $(U(1)_R)$ $N=2$ MESGT AND YMESGTs WITH FLAT POTENTIALS IN $D=4$ (1984):
Cremmer, Derendinger, de Wit, Ferrara, Girardello, Kounnas, van Proeyen

A "NATURAL" FAMILY OF $N=2$ MESGTs THAT ALLOW GAUGINGS WITH FLAT POTENTIALS ARE THOSE THEORIES THAT DESCEND FROM $D=5$ MESGTs DEFINED BY JORDAN ALGEBRAS OF DEGREE THREE.

CLASSIFICATION OF $D=4$, $N=2$, MESGTs WITH SCALAR MANIFOLDS THAT ARE SYMMETRIC SPACES (Cremmer & van Proeyen

NON-JORDAN FAMILY : $\mathcal{M} = \frac{SU(N,1)}{U(N)}$

DO NOT HAVE A 5D ORIGIN FOR $N>1$.

+ $N=2$ MESGTs DEFINED BY JORDAN ALGEBRAS WHICH CAN BE OBTAINED FROM 5D BY DIM. REDUCTION.

NOTE THAT THE 5-DIMENSIONAL CORRESPONDENCE BETWEEN VECTOR FIELDS A_μ^I AND ELEMENTS e_I OF THE UNDERLYING JORDAN ALGEBRA J GETS EXTENDED TO THE CORRESPONDENCE

$$\begin{pmatrix} W^0_{\mu\nu} & F^I_{\mu\nu} \\ \tilde{F}^I_{\mu\nu} & \tilde{W}^0_{\mu\nu} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \alpha & J \\ \tilde{J} & \beta \end{pmatrix} \equiv \mathcal{F}(J)$$

$A^0_\mu \leftrightarrow g_{\mu 5}$ $\qquad$ $\alpha, \beta \in \mathbb{R}$

$\mathcal{F}(J)$ = Freudenthal triple system defined over a Jordan algebra of degree 3

4D U-DUALITY GROUP $\equiv$ AUTOMORPHISM GROUP OF $\mathcal{F}(J)$

$$\mathcal{M} = \frac{Aut(\mathcal{F}(J))}{Ktr(J)}$$

$$Z^I = A^I + i e^\sigma h^I$$

$$A^I = A^I_5$$

GST83

Kähler Potential :

$$K(z,\bar{z}) = -\ln N(z-\bar{z})$$

$$N(z-\bar{z}) = C_{IJK}(z^I-\bar{z}^I)(z^J-\bar{z}^J)(z^K-\bar{z}^K)$$

6D MINIMAL SUGRA COUPLED TO TENSOR AND VECTOR MULTIPLETS

6D origin of the N=2 MESGTs defined by Jordan algebras of degree 3 was first studied by Larry Romans.

Vector multiplets do not have scalars in D=6. $(A_\mu, \lambda^i(L))$.

Tensor multiplet has one real scalar

$$B^+_{\mu\nu}, \ \lambda^i(R), \ \phi$$

Minimal sugra coupled to n_T tensor multiplets has the scalar manifold

$$\mathcal{M}_6^T = \frac{SO(n_T,1)}{SO(n_T)}$$

Under dimensional reduction they yield the generic Jordan family of N=2 MESGTs in D=5 with manifolds

$$\frac{SO(n_T,1)}{SO(n_T)} \times SO(1,1)$$

D=6 tensor multiplet → D=5 vector multiplet

Romans suggested that one might be able to obtain the magical N=2 MESGT's from D=6 by coupling vector multiplets to tensor coupled sugra as follows:

$$D=6 \longrightarrow D=5$$

$$n_T = 2\,,\quad n_V = 2 \longrightarrow J_3^{\mathbb{R}} \text{ MESGT}$$

$$n_T = 3\,,\quad n_V = 4 \longrightarrow J_3^{\mathbb{C}} \text{ MESGT}$$

$$n_T = 5\,,\quad n_V = 8 \longrightarrow J_3^{\mathbb{H}} \text{ MESGT}$$

$$n_T = 9\,,\quad n_V = 16 \longrightarrow J_3^{\mathbb{O}} \text{ MESGT}$$

$$J_2^{\mathbb{A}} \oplus \begin{pmatrix} 0 & 0 & q_1 \\ 0 & 0 & q_2 \\ \bar{q}_1 & \bar{q}_2 & 0 \end{pmatrix} \longrightarrow J_3^{\mathbb{A}}$$

$$q_1, q_2 \in \mathbb{A} \qquad \mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$$

Subsequent work on $D=6$ minimal sugra coupled to tensors and vectors confirmed Romans idea! (Sezgin + Nishino; Sagnotti, Ferrara, Riccioni, Minasian, ...)

In fact, the basis used by de Wit + van Proeyen in their classification of homogeneous manifolds of $D=5$ MESGTs is similar in spirit and makes the connection to $D=6$ manifest!

SCHERK-SCHWARZ GENERALIZED DIMENSIONAL REDUCTION $\Rightarrow$ SPONTANEOUSLY BROKEN N=8 SUPERGRAVITY IN D=4 WITH FLAT POTENTIAL.

Scherk, Schwarz (79), Cremmer, S, S (79)

4D interpretation of the resulting theory as gauged supergravity with "flat" gauge algebra

(Andrianopoli, D'Auria, Ferrara, Lledo (2002)

$$E_{7(7)} = \tilde{27} \oplus E_{6(6)} \times SO(1,1) \oplus 27$$

$$USp(8) \downarrow \underbrace{U(1) \oplus 27}_{\text{Flat gauge algebra}}$$

NO SCALE SUGRA THEORIES WITH N=2 OBTAINED BY SCHERK-SCHWARZ MECHANISM WITH FLAT GAUGE GROUPS IN D=5 AND D=4

NO SCALE SUPERGRAVITY THEORIES DEFINED BY JORDAN ALGEBRAS ARE SPECIAL.

ADFL, Trigiante

M/STRING THEORETIC ORIGINS OF $N=2$ MESGT'S DEFINED BY JORDAN ALGEBRAS, IN PARTICULAR THE MAGICAL THEORIES ?

DUAL PAIRS OF TYPE II STRING COMPACTIFICATION AS CONSTRUCTED BY SEN & VAFA (1995) USING METHODS DEVELOPED BY VAFA & WITTEN.

ORBIFOLDING $T^4 \times S^1 \times S^1 / \Gamma$ + ADIABATIC ARGUMENT

$N=2$, $d=4$ THEORY WITH 15 MASSLESS VECTOR MULTIPLETS AND NO HYPERS

$\Rightarrow$ $J_3^{\mathbb{H}}$ MODEL WITH $\mathcal{M}_4 = \dfrac{SO^*(12)}{U(6)}$

SELF-DUAL (DILATON IN VECTOR MULTIPLET.)

- GENERIC JORDAN THEORY WITH 7 VECTOR MULT.

$$\mathcal{M}_4 = \frac{SO(6,2) \times SU(1,1)}{SO(6) \times U(1) \times U(1)}$$

NOT SELF-DUAL

- STU MODEL COUPLED TO 4 HYPERS

$$\mathcal{M}_4 = \left[\frac{SU(1,1)}{U(1)}\right]^3 \times \frac{SO(4,4)}{SO(4) \times SO(4)}$$

SELF-DUAL

- $N=6$ THEORY WITH $\mathcal{M}_4 = \dfrac{SO^*(12)}{U(6)}$

MODULI SPACE SAME AS $J_3^{\mathbb{H}}$ THEORY

SELF-DUAL

M/STRINGY ORIGIN OF THE EXCEPTIONAL SUGRA WITH $\mathcal{M}_4 = \dfrac{E_{7(-25)}}{E_6 \times U(1)}$?

FHSV MODEL :

TYPE II ST ON A SELF-MIRROR CY WITH $h^{11} = h^{12} = 11$ DUAL TO HETEROTIC ST ON $K3 \times T^2$

QUANTUM MODULI ≡ CLASSICAL MODULI

$$\mathcal{M}_4 = \mathcal{M}_V \times \mathcal{M}_H$$

$$\mathcal{M}_V = \frac{SO(10,2) \times SU(1,1)}{SO(10) \times U(1) \times U(1)} \qquad \mathcal{M}_H = \frac{SO(12,4)}{SO(12) \times SO(4)}$$

→ MAXIMAL GENERIC JORDAN THEORY THAT SITS INSIDE THE EXCEPTIONAL MESGT.

$$SO(10,2) \times SU(1,1) \subset E_{7(-25)} \quad \& \quad \not\subset E_{7(7)}$$

THERE EXISTS A UNIQUE ANOMALY-FREE SUGRA IN $d=6$ THAT REDUCES TO THE EXCEPTIONAL $N=2$ MESGT COUPLED TO 28 HYPERMULTIPLETS ! MG & SEZGIN

$$\mathcal{M}_4 = \frac{E_{7(-25)}}{E_6 \times U(1)} \times \frac{E_{8(-24)}}{E_7 \times SU(2)} = \mathcal{M}_V \times \mathcal{M}_H$$

$$\mathcal{M}_3 = \frac{E_{8(-24)}}{E_7 \times SU(2)} \times \frac{E_{8(-24)}}{E_7 \times SU(2)}$$

MODULI SPACE OF FHSV MODEL IS THE SUBSPACE

$$\frac{SO(12,4)}{SO(12) \times SO(4)} \times \frac{SO(12,4)}{SO(12) \times SO(4)}$$

WORK OF BENEDICT GROSS :

U-DUALITY GROUPS OF THE EXCEPTIONAL MESGT ADMIT ARITHMETIC FORMS OVER $\mathbb{Z}$

$$E_{6(-26)}(\mathbb{Z}) \ , \ E_{7(-25)}(\mathbb{Z}) \ , \ E_{8(-24)}(\mathbb{Z})$$

INDECOMPOSABLE INTEGRAL STRUCTURE OVER $J_3^{\mathbb{O}}(\mathbb{Z})$ $\overset{?}{\Longleftrightarrow}$ Some exceptional Calabi_Yau

Exceptional Tube Domain $E_{7(-25)} / E_6 \times U(1)$ $\Longleftrightarrow$ Deformations of Hodge structures of the exceptional CY

ALL THE ABOVE FACTS SUGGEST STRONGLY THAT AN EXCEPTIONAL CALABI-YAU MANIFOLD $CY_3^{\mathbb{O}}$ EXISTS SUCH THAT M-THEORY OVER $CY_3^{\mathbb{O}}$ LEADS TO THE EXCEPTIONAL MESGT DEFINED BY $J_3^{\mathbb{O}}$ COUPLED TO HYPERMULTIPLETS WITH $\mathcal{M}_Q = E_{8(-24)} / E_7 \times SU(2)$

ALL SIMPLE LIE ALGEBRAS INCLUDING G_2, F_4, E_8 ADMIT A 5-GRADED DECOMPOSITION w.r.t. A SUBALGEBRA g^0 OF MAXIMAL RANK

$$g = g^{-2} \oplus g^{-1} \oplus g^{0} \oplus g^{+1} \oplus g^{+2} , \qquad \dim g^{\pm 2} = 1$$

THEY CAN BE REALIZED OVER FREUDENTAL-KANTOR TRIPLE SYSTEMS $\mathcal{F} \Longleftrightarrow$ SUBSPACE g^{+1}

FREUDENTHAL'S CONSTRUCTION OF F_4, E_6, E_7, E_8

$$\mathcal{F}(J_3^A) = \begin{pmatrix} \alpha & J_3^A \\ \tilde{J}_3^A & \beta \end{pmatrix}$$

FREUDENTHAL TRIPLE PRODUCT:

$$(X, Y, Z) \in \mathcal{F}(J)$$

$$\forall \quad X, Y, Z \in \mathcal{F}(J)$$

$A = \mathbb{R} \longleftrightarrow F_4$

$A = \mathbb{C} \longleftrightarrow E_6$

$A = \mathbb{H} \longleftrightarrow E_7$

$A = \mathbb{O} \longleftrightarrow E_8$

Automorphism Groups of $\mathcal{F}(J_3^A)$

$$\mathrm{Aut}\, \mathcal{F}(J_3^{O_s}) \approx E_{7(7)}$$

$$\mathrm{Aut}\, \mathcal{F}(J_3^{O}) \approx E_{7(-25)}$$

$$\mathrm{Aut}\, \mathcal{F}(J_3^{\mathbb{H}}) \approx SO^*(12)$$

$$\mathrm{Aut}\, \mathcal{F}(J_3^{\mathbb{C}}) \approx SU(3,3)$$

$$\mathrm{Aut}\, \mathcal{F}(J_3^{\mathbb{R}}) \approx Sp(6,\mathbb{R})$$

$$\mathrm{Aut}\, \mathcal{F}(\mathbb{R} + \Gamma(d)) \approx SO(d,2) \times SO(2,1)$$

$\mathcal{F}$ ADMITS A SYMPLECTIC INVARIANT FORM

$$\langle X, Y \rangle = - \langle Y, X \rangle \qquad X, Y \in \mathcal{F}$$

SUCH THAT THE QUARTIC INVARIANT I_4 OF $\mathcal{F}$ IS:

$$I_4(X) \equiv \langle (X, X, X), X \rangle$$

NOTE THAT $\mathrm{Aut}(\mathcal{F}(J_3)) \cong$ Conformal Group of J_3

QUADRATIC MAP : $\bar{X} \longrightarrow X^{\#}$

$$X^{\#}_I = C_{IJK} X^J X^K$$

ADJOINT IDENTITY : $(X^{\#})^{\#} = \mathcal{N}(X)\, X$

IMPLIES THAT $\mathcal{N}(X) = C_{IJK} X^I X^J X^K$ IS THE NORM FORM OF A JORDAN ALGEBRA OF DEGREE THREE.

$\Longleftrightarrow$ $F(X) = \dfrac{\mathcal{N}(X)}{X^0}$ IS LEGENDRE INVARIANT

MG + PIOLINE

$\Longleftrightarrow$ ONE-TO-ONE CORRESPONDENCE WITH THE LIST OF LEGENDRE INVARIANT CUBIC POLYNOMIALS BY KAZHDAN et. al (2000).

PHYSICAL MEANING OF THE ADJOINT IDENTITY FROM A SPACE-TIME POINT OF VIEW :

ADJOINT IDENTITY FOR THE 4 SIMPLE JORDAN ALGEBRAS OF DEGREE THREE IS EQUIVALENT TO THE FIERZ IDENTITIES FOR THE EXISTENCE OF SUPERSYMMETRIC YANG-MILLS THEORIES IN $d = 3, 4, 6, 10$

e.g

$E_{6(-26)} \supset SO(9,1)$

$J_3^{\mathbb{O}} = 1 + 10 + 16$

$$\left(\begin{array}{cc|c} \alpha & o_1 & o_2 \\ \bar{o}_1 & \beta & o_3 \\ \hline \bar{o}_2 & \bar{o}_3 & \gamma \end{array}\right)$$

Sierra

THE ABOVE FIERZ IDENTITIES EXTEND TO SOME REMARKABLE FIERZ IDENTITIES IN 5, 6, 8 AND 12 DIMENSIONS BY GOING FROM THE JORDAN ALGEBRA TO THE CORRESPONDING FREUDENTHAL TRIPLE SYSTEMS

M.G + Pavlyk

$\mathcal{F}(J_3^{\mathbb{O}}) \cong 56$ of $Aut(\mathcal{F}) = E_{7(-25)}$

$E_{7(-25)} \supset SL(2,\mathbb{R}) \times SO(10,2)$

$56 = (2,12) + (1,32)$

SOME FURTHER APPLICATIONS OF JORDAN ALGEBRAS AND FREUDENTHAL TRIPLE SYSTEMS

- Classification of orbits of BPS BH solutions under the action of U-duality groups in $N=2$ MESGT's with symmetric target spaces & $N=8$ sugra in $d=5$ & 4

  Ferrara & M.G.

- 4D U-duality groups realized as spectrum generating conformal groups of the underlying Jordan algebra in 5D MESGTs and in $N=8$ sugra. (FG, M.G, Koepsell & Nicolai)

- 3D U-duality groups realized as spectrum generating quasiconformal groups in 4D MESGT's and $N=8$ GKN

  M.G & Pavlyk

  & construction of the minimal unitary representations of U-duality groups

- Holomorphic anomalies & heat equations

  M.G, Neitzke & Pioline

- Attractor flows, BPS black hole degeneracies and automorphic forms GNP & Waldron

- Non-BPS attractors and their orbits in $N=2$ MESGTs, Bellucci, Ferrara, M.G, Marrani