U-DUALITY AS CONFORMAL AND QUASICONFORMAL SYMMETRY, EXTREMAL BLACK HOLES AND REPRESENTATION THEORY

MURAT GÜNAYDIN

CERN BH INSTITUTE

SEPT. 24, 2008

TOPICS

- REVIEW OF N=2 MAXWELL-EINSTEIN SUPERGRAVITY THEORIES WITH SCALAR MANIFOLDS $\mathcal{M}_5$
- UNIFIED MESGT'S IN $d=5$.
- SYMMETRIC $\mathcal{M}_5 \Longleftrightarrow$ EUCLIDEAN JORDAN ALGEBRAS OF DEGREE 3
- UNIFIED MESGT'S WITH $\mathcal{M}_5$ NON-HOMOGENEOUS $\Leftarrow$ LORENTZIAN JORDAN ALGEBRAS OF ARBITRARY DEGREE $\neq 4$
- STRINGY ORIGINS OF N=2 MESGT'S DEFINED BY JORDAN ALGEBRAS
- ORBITS OF EXTREMAL 5d BLACK HOLES (BPS + NON-BPS) UNDER U DUALITY.
- 4d U-DUALITY GROUPS AS SPECTRUM GENERATING CONFORMAL GROUPS IN 5 DIMENSIONS
- N=2 MESGT'S IN $d=4$ AND FREUDENTHAL TRIPLE SYSTEMS DEFINED BY JORDAN ALGEBRAS
- ORBITS OF 4d EXTREMAL BLACK HOLES UNDER U-DUALITY.
- 3d U-DUALITY GROUPS AS SPECTRUM GENERATING QUASI-CONFORMAL GROUPS IN 4 DIMENSIONS
- MINIMAL UNITARY REPRESENTATIONS OF NON-COMPACT GROUPS FROM QUANTIZATION OF THEIR GEOMETRIC REALIZATIONS AS QUASICONFORMAL GROUPS THAT LEAVE A "LIGHT-CONE" WITH RESPECT TO A QUARTIC DISTANCE FUNCTION.
- QUANTUM ATTRACTOR FLOWS FOR SSS 4d, BPS BLACK HOLES AND UNITARY REPS. OF QConf(J)
- HARMONIC SUPERSPACE, N=2 SIGMA MODELS AND MINIMAL UNITARY REPRESENTATIONS.
- CONCLUSIONS + OPEN PROBLEMS

VERY SPECIAL GEOMETRY OF 5d, N=2 SUPERGRAVITY COUPLED TO N=2 VECTOR MULTIPLETS: MG, SIERRA + TOWNSEND (1983)

("very special real geometry")

$$\left(e_\mu^m + \psi_\mu^i + \underbrace{A_\mu^0 \right) \oplus \left(A_\mu^a}_{A_\mu^I} + \lambda^{ai} + \phi^a \right)$$

$$e^{-1}\mathcal{L}_{\text{bosonic}} = -\tfrac{1}{2}R - \tfrac{1}{4}\mathring{a}_{IJ} F_{\mu\nu}^I F^{J\mu\nu} - \tfrac{1}{2} g_{xy} \partial_\mu \varphi^x \partial^\mu \varphi^y$$
$$+ \frac{e^{-1}}{6\sqrt{6}} C_{IJK} F_{\mu\nu}^I F_{\lambda\rho}^J A_\sigma^K \epsilon^{\mu\nu\lambda\rho\sigma}$$

$a, b = 1, \ldots n$

$I, J, \ldots = 0, 1, \ldots, n$

$x, y, \ldots = 1, \ldots, n$

CONSTANT SYMMETRIC TENSOR C_{IJK}

REMARKABLE FACT: N=2 Maxwell-Einstein Supergravity (MESGT) IS UNIQUELY DETERMINED BY C_{IJK}

SCALAR MANIFOLD $\mathcal{M}$ IS AN HYPERSURFACE IN AN $(n+1)$ DIMENSIONAL AMBIENT SPACE $\mathcal{C}$ WITH COORDINATES h^I AND METRIC a_{IJ}:

$$a_{IJ} = -\tfrac{1}{3} \partial_I \partial_J \ln \mathcal{N}(h)$$

$$\mathcal{N}(h) = C_{IJK} h^I h^J h^K$$

$\mathcal{M}$ is THE $\mathcal{N}(h) = 1$ HYPERSURFACE

$$\mathring{a}_{IJ}(\varphi) = a_{IJ}(h)\Big|_{\mathcal{N}=1}$$

$$g_{xy} = \mathring{a}_{IJ} h_x^I h_y^J \qquad h_x^I \equiv \frac{\partial h^I}{\partial \varphi^x}$$

GLOBAL SYMMETRIES $\equiv$ SYMMETRIES OF C_{IJK}

ELEVEN DIMENSIONAL SUGRA COMPACTIFIED OVER A CY 3FOLD YIELDS A 5d N=2 MESGT COUPLED TO N=2 HYPERMULTIPLETS

C_{IJK} = TOPOLOGICAL INTERSECTION NUMBERS OF CY

$h_{1,1} = n_V + 1 \qquad h_{2,1} = n_H - 1$

RIEMANN TENSOR OF SCALAR MANIFOLD $\mathcal{M}$ OF N=2 MESGT

$$R_{xyzu} = \frac{4}{3}\left\{ g_{x[u}\, g_{z]y} + T_{x[u}{}^{w}\, T_{z]yw} \right\}$$

$$T_{xyz} = C_{IJK}\, h^I_x\, h^J_y\, h^K_z$$

IF T_{xyz} IS COVARIANTLY CONSTANT THEN $\mathcal{M}$ IS A SYMMETRIC SPACE

$$T_{xyz;u} = 0 \implies R_{xyzu;w} = 0$$

$T_{xyz;u} = 0 \oplus$ N=2 SUSY $\oplus$ POSITIVITY OF K.E IMPLY THAT $\mathcal{N}(h) = C_{IJK}\, h^I h^J h^K$ CAN BE IDENTIFIED WITH THE NORM FORM OF A EUCLIDEAN JORDAN ALGEBRA J OF DEGREE 3 !

$$\mathcal{M} = \frac{Str_0(J)}{Aut(J)}$$

$Str_0(J)$ = INVARIANCE GROUP OF THE NORM OF J
≡ REDUCED STRUCTURE GROUP OF J

$Aut(J)$ = AUTOMORPHISM GROUP OF J

JORDAN ALGEBRAS

$$X \circ Y = Y \circ X \qquad X, Y \in J$$

$$X \circ (Y \circ X^2) = (X \circ Y) \circ X^2$$

EUCLIDEAN (FORMALLY REAL) JORDAN ALGEBRAS HAVE THE PROPERTY

$$X^2 + Y^2 = 0 \implies X = 0 \text{ AND } Y = 0$$

$n \times n$ HERMITIAN MATRICES OVER $A = \mathbb{R}, \mathbb{C}, \mathbb{H}$ FORM A JORDAN ALGEBRA J_n^A WITH THE PRODUCT

$$X \circ Y = \frac{1}{2}(XY + YX)$$

WHICH PRESERVES HERMITICITY.

COMPLETE LIST OF SIMPLE EUCLIDEAN JORDAN ALGEBRAS (FINITE DIMENSIONAL):

$$J_n^{\mathbb{R}}, \quad J_n^{\mathbb{C}}, \quad J_n^{\mathbb{H}}$$

$J_3^{\mathbb{O}}$ = 3x3 Hermitian over real octonions

$\Gamma(D) \equiv$ DIRAC GAMMA MATRICES IN D-DIM'NAL EUCLIDEAN SPACE

$J_3^{\mathbb{O}}$ HAS NO REALIZATION IN TERMS OF ASSOCIATIVE MATRICES WITH THE ABOVE PRODUCT !

$\iff$ EXCEPTIONAL JORDAN ALGEBRA

$\exists$ FOUR SIMPLE JORDAN ALGEBRAS OF DEGREE 3

$$J_3^{\mathbb{R}}, \quad J_3^{\mathbb{C}}, \quad J_3^{\mathbb{H}}, \quad J_3^{\mathbb{O}}$$

AND AN INFINITE FAMILY OF NON-SIMPLE JORDAN ALGEBRAS OF DEGREE 3

$$J = \mathbb{R} \oplus \Gamma(D)$$

GENERALIZED SPACETIMES COORDINATIZED BY JORDAN ALGEBRAS Nuovo Cimento 1975

MINKOWSKI SPACE: COORDINATES x_μ, METRIC $\eta_{\mu\nu}$

$x = x_\mu \sigma^\mu$ $\sigma^\mu = (1, \sigma^i)$ $i = 1,2,3$, Pauli Matrices

$x \in J_2^{\mathbb{C}}$ $x \circ y \equiv \frac{1}{2}(xy + yx) \in J_2^{\mathbb{C}}$

$\mathrm{Aut}(J_2^{\mathbb{C}}) = SU(2)$

$N(x) = \mathrm{Det}\, x = \eta_{\mu\nu} x^\mu x^\nu$, $\eta_{\mu\nu} = (+---)$

$\mathrm{Str}_0(J_2^{\mathbb{C}}) = SL(2,\mathbb{C})$ = INVARIANCE GROUP OF NORM

$\mathrm{Möb}(J_2^{\mathbb{C}}) = SU(2,2)$ = CONFORMAL GROUP

Möbius group ≡ linear fractional group (generated by (translations) + (inversions) + ($\mathrm{Str}(J) \equiv \mathrm{Str}_0(J) \times \mathcal{D}$)

FOR A GENERAL JORDAN ALGEBRA J WITH A BASIS e_I ($I = 1, \ldots, \dim(J)$) AND NORM FORM N

$x = e_I x^I$ $\|x\| = N(x)$

$\mathrm{Aut}(J) \equiv$ Rotation Group (RG)

$\mathrm{Str}_0(J) \equiv$ Lorentz Grp (LG) (invariance group of N)

$\mathrm{Möb}(J) \equiv$ Conformal Group (CG)

EXAMPLE: EXCEPTIONAL JORDAN ALGEBRA

	OVER REAL OCTONIONS $J_3^{\mathbb{O}}$	OVER SPLIT OCTONIONS $J_3^{\mathbb{O}_s}$
RG	F_4	$F_{4(4)}$
LG	$E_{6(-26)}$	$E_{6(6)}$
CG	$E_{7(-25)}$	$E_{7(7)}$

SYMMETRIES OF GEN. SPACETIMES DEFINED BY EUCLIDEAN (FORMALLY REAL) JORDAN ALGEBRAS J

J	$RG(J)$	$LG(J)$	$Conf(J)$
$\Gamma(d)$	$SO(d)$	$SO(d,1)$	$SO(d,2)$
$J_n^{\mathbb{R}}$	$SO(n)$	$SL(n,\mathbb{R})$	$Sp(2n,\mathbb{R})$
$J_n^{\mathbb{C}}$	$SU(n)$	$SL(n,\mathbb{C})$	$SU(n,n)$
$J_n^{\mathbb{H}}$	$USp(2n)$	$SU^*(2n)$	$SO^*(4n)$
$J_3^{\mathbb{O}}$	F_4	$E_{6(-26)}$	$E_{7(-25)}$

J_n^{A} = JA of $n \times n$ Hermitian matrices over the division algebra A

$\Gamma(d)$ = Dirac gamma matrices in d dimensions

CONFORMAL GROUPS OF EUCLIDEAN JORDAN ALGEBRAS ALL ADMIT POSITIVE ENERGY UNITARY REPRESENTATIONS. $\Rightarrow$ CAUSAL SPACETIMES

CONFORMAL GROUP OF A JORDAN ALGEBRA J OF DEGREE p ACTS AS A "LINEAR FRACTIONAL GROUP" OF J. KOECHER

NORM FORM N INDUCES A SYMMETRIC p-FORM

$$\rightarrow \quad N(X) \equiv N(\underbrace{X, \ldots, X}_{p}) \quad \text{OVER } J.$$

Conf(J) LEAVES INVARIANT THE "MEASURE OF THE p ANGLE" FORMED BY p "STRAIGHT" LINES WITH DIRECTION VECTORS $\underbrace{X, Y, \ldots, Z}_{p}$

$$\frac{N(X,Y,\ldots,Z)^p}{N(X)\cdots\cdots N(Z)}$$

KANTOR

CONFORMAL GROUP OF J LEAVES INVARIANT THE CROSS RATIO

$$\frac{N(\bar{x}-\bar{z})\, N(\bar{y}-\bar{w})}{N(\bar{y}-\bar{z})\, N(\bar{x}-\bar{w})}$$

KANTOR

FOR $p=2$ THEY REDUCE TO THE STANDARD PROPERTIES OF CONFORMAL TRANSFORMATIONS.

THE LIE ALGEBRA $g(J)$ OF $\mathrm{Conf}(J)$ HAS A 3-GRADED DECOMPOSITION

$$g(J) = g^{-1} \oplus g^{0} \oplus g^{+1}$$

g^0 = structure algebra of J

$g^{-1} \Leftrightarrow J \quad + \quad g^{+1} \Leftrightarrow \bar{J}$

$$g(J) = U_a \oplus S_{ab} \oplus \tilde{U}_b \qquad a, b \in J$$

$$[U_a, \tilde{U}_b] = S_{ab} \in g^0$$

$$[S_{ab}, U_c] = U_{\{abc\}} \in g^{-1}$$

$$[S_{ab}, \tilde{U}_c] = \tilde{U}_{\{bac\}} \in g^{+1}$$

Tits
Kantor
Koecher

$$[S_{ab}, S_{cd}] = S_{\{abc\}d} - S_{\{bad\}c}$$

$$\{abc\} \equiv a\circ(b\circ c) + c\circ(b\circ a) - (a\circ c)\circ b \qquad \text{JTP}$$

$$\{abc\} = \{cba\}$$

CONFORMAL REALIZATION: $x \in J$

$U_a(x) = a$ TRANSLATIONS

$S_{ab}(x) = \{abx\}$ LORENTZ GROUP × DILATATIONS

$\tilde{U}_c(x) = -\frac{1}{2}\{xcx\}$ SPECIAL CONFORMAL TRANSFORMATIONS

STRUCTURE GROUP = LORENTZ GROUP × DILATIONS

POINCARÉ GROUP OF J ≡ $\mathrm{STR}(J) \circledS$ Translations

QUADRATIC MAP : $X \longrightarrow X^{\#}$

$$(X^{\#})_I = C_{IJK} X^J X^K$$

$$T_{xyz;u} = 0 \Rightarrow (X^{\#})^{\#} = N(X)\, X$$ ADJOINT IDENTITY

$$N(X) = C_{IJK} X^I X^J X^K$$ cubic norm of a degree 3 Jordan Algebra

SCALAR MANIFOLD of N=2 MESGT :
$$\mathcal{M} = \frac{Str_0(J)}{Aut(J)}$$

ADJOINT IDENTITY $\Longleftrightarrow$ LEGENDRE INVARIANT CUBIC POLYNOMIALS Kazhdan & Etingof Polischuk (2000)

GENERIC JORDAN FAMILY : $J = \mathbb{R} \oplus \Gamma(Q)$

$$\mathcal{N} = \alpha Q$$ $\alpha \in \mathbb{R}$, Q is a quadratic form of signature $(+,-,-,\cdots)$

$$\mathcal{M} = \frac{SO(n-1,1) \times SO(1,1)}{SO(n-1)}$$

FOR N=2 MESGT's DEFINED BY SIMPLE JORDAN ALGEBRAS OF DEGREE 3 :

$$\mathcal{M}(J_3^{\mathbb{R}}) = \frac{SL(3,\mathbb{R})}{SO(3)}$$ (5+1) vectors

$$\mathcal{M}(J_3^{\mathbb{C}}) = \frac{SL(3,\mathbb{C})}{SU(3)}$$ (8+1) vectors

$$\mathcal{M}(J_3^{\mathbb{H}}) = \frac{SU^*(6)}{USp(6)}$$ 14+1 vectors

$$\mathcal{M}(J_3^{\mathbb{O}}) = \frac{E_{6(-26)}}{F_4}$$ 26+1 vectors

NORM FORM $\cong$ 'DETERMINANT' OF J_3^A

N=2, 5d MESGT's DEFINED BY SIMPLE JORDAN ALGEBRAS OF DEGREE 3 ARE UNIFIED THEORIES

$\longleftarrow$ SUSY $\longrightarrow$

$$e_\mu^m \quad \psi_\mu^i \quad \begin{pmatrix} A_\mu^0 \\ A_\mu^a \end{pmatrix} \updownarrow G \quad \lambda^{ia} \quad \phi^a \qquad\qquad \mathcal{M} = G/H$$

$\longleftarrow$ SUSY $\longrightarrow$

UNIFIED MESGT $\Longleftrightarrow$ C_{IJK} $\equiv$ INVARIANT TENSOR OF A SIMPLE GROUP G

THEY ARE THE ONLY UNIFIED MESGT'S IN $d=5$ WHOSE SCALAR MANIFOLDS ARE SYMMETRIC SPACES !

GST 1983

FURTHERMORE THERE EXIST 3 INFINITE FAMILIES OF UNIFIED MESGT'S IN $d=5$ WHOSE SCALAR MANIFOLDS ARE NOT HOMOGENEOUS SPACES (plus a sporadic one)

M.G + Zagermann 2003

C_{IJK} $\equiv$ STRUCTURE CONSTANTS OF NONCOMPACT (LORENTZIAN) JORDAN ALGEBRAS OF DEGREE > 4

$$J^{\mathbb{R}}_{(1,N)_0}, \quad J^{\mathbb{C}}_{(1,N)_0}, \quad J^{\mathbb{H}}_{(1,N)_0}, \quad N > 3$$

(plus $J^{\mathbb{O}}_{(1,2)_0}$)

GENERATED BY MATRICES OVER $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ THAT ARE HERMITIAN WITH RESPECT TO A LORENTZIAN METRIC

$$(X\eta)^\dagger = X\eta \qquad\qquad \eta = \mathrm{Diag}(+,-,-,\dots)$$

UNIFIED $N=2$ MESGT'S IN FIVE DIMENSIONS DEFINED BY LORENTZIAN JORDAN ALGEBRAS AND THEIR SYMMETRY GROUPS G $(N \neq 3)$

J	$G = Aut(J)$	# VECTOR FIELDS
$J^{\mathbb{R}}_{(1,N)_0}$	$SO(N,1)$	$\frac{1}{2}N(N+3)$
$J^{\mathbb{C}}_{(1,N)_0}$	$SU(N,1)$	$N(N+2)$
$J^{\mathbb{H}}_{(1,N)_0}$	$USp(2N,2)$	$N(2N+3)$
$J^{\mathbb{O}}_{(1,2)_0}$	$F_{4(-20)}$	26

$C_{IJK} \equiv d_{IJK} \equiv$ structure constants of the traceless elements of $J^{\mathbb{A}}_{(1,N)}$

IN THE 5d, $N=2$, MESGT'S DEFINED BY EUCLIDEAN JORDAN ALGEBRAS OF DEGREE 3 C_{IJK} ARE GIVEN BY THE NORM FORM!

$N=2$ MESGT'S DEFINED BY $J_3^{\mathbb{C}}$, $J_3^{\mathbb{H}}$, $J_3^{\mathbb{O}}$ ARE EQUIVALENT TO THOSE DEFINED BY $J^{\mathbb{R}}_{(1,3)_0}$, $J^{\mathbb{C}}_{(1,3)_0}$ AND $J^{\mathbb{H}}_{(1,3)_0}$.

OF THE 3 INFINITE FAMILIES ONLY THE FAMILY DEFINED ALGEBRAS $J^{\mathbb{C}}_{(1,N)_0}$ CAN BE GAUGED SO AS TO OBTAIN AN INFINITE FAMILY OF UNIFIED YANG-MILLS-EINSTEIN SUPERGRAVITY THEORIES WITH GAUGE GROUPS $SU(N,1)$.

GEOMETRIES OF THE THEORIES DEFINED BY LORENTZIAN JORDAN ALGEBRAS?

GEOMETRIES OF 5d MESGT'S DEFINED BY JORDAN ALGEBRAS $J^{\mathbb{A}}_{(1,2)_0}$ $(\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O})$ AND CARTAN'S REMARKABLE ISOPARAMETRIC HYPERSURFACES :

$$J = \begin{pmatrix} -\xi^0 + \xi^4 & \xi^3 & -\bar{\xi}^2 \\ \bar{\xi}^3 & -\xi^0 - \xi^4 & -\xi^1 \\ \xi^2 & \bar{\xi}^1 & 2\xi^0 \end{pmatrix} \in J^{\mathbb{A}}_{(1,2)_0} , \quad \begin{matrix} \xi \in \mathbb{A} , \\ i = 1,2,3 \\ , \ \xi \ \xi \in \mathbb{R} \end{matrix}$$

Scalar manifold of $N=2$ MESGT in $d=5$ is given by the hypersurface :

$$N(\xi) = d_{IJK} \xi^I \xi^J \xi^K = \frac{1}{3} \operatorname{Tr} J^3 = 1$$

$$\xi^I \Big|_{N=1} = h^I(\varphi^x)$$

$\operatorname{Aut}(J^{\mathbb{R}}_{(1,2)}) = SO(2,1)$, $\operatorname{Aut}(J^{\mathbb{C}}_{(1,2)}) = SU(2,1)$

$\operatorname{Aut}(J^{\mathbb{H}}_{(1,2)}) = USp(6,2)$, $\operatorname{Aut}(J^{\mathbb{O}}_{(1,2)}) = F_{4(-20)}$

THE SCALAR MANIFOLDS OF THESE THEORIES ARE FOLIATED BY THE NON-COMPACT ANALOGS OF CARTAN'S ISOPARAMETRIC HYPERSURFACES.

CARTAN'S HYPERSURFACES ARE DETERMINED BY THE CUBIC FUNCTION $\mathcal{F}$ DEFINED OVER EUCLIDEAN JORDAN ALGEBRAS: $\mathcal{F} = \frac{1}{3} \operatorname{Tr}(J^3)$, $J \in (J^{\mathbb{A}}_3)_0$

IN 4, 7, 13 AND 25 DIMENSIONAL SPHERES DEFINED BY THE CONDITION

$$\operatorname{Tr} J^2 = \text{constant}.$$

FOR LORENTZIAN JORDAN ALGEBRAS THE CORRESPONDING CONDITIONS ARE IN SPACES OF SPLIT SIGNATURES:

$(3,2)$, $(4,4)$, $(6,8)$, $(10,16)$

$\Downarrow$

AdS_4

SCALAR MANIFOLDS OF $N=2$ MESGT'S DEFINED BY $J^{\mathbb{R}}_{(1,3)_0}$, $J^{\mathbb{C}}_{(1,3)_0}$, $J^{\mathbb{H}}_{(1,3)_0}$ ARE SYMMETRIC SPACES

$$J^{\mathbb{R}}_{(1,3)_0} \Rightarrow \frac{SL(3,\mathbb{C})}{SU(3)}$$

$$J^{\mathbb{C}}_{(1,3)_0} \Rightarrow \frac{SU^*(6)}{USp(6)}$$

$$J^{\mathbb{O}}_{(1,3)_0} \Rightarrow \frac{E_{6(-26)}}{F_4}$$

FOR $N>3$, THE SCALAR MANIFOLDS OF THE THEORIES DEFINED BY $J^{\mathbb{A}}_{(1,N)_0}$ $(\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H})$ ARE "CUBIC SUBMANIFOLDS" OF COHOMOGENEITY 2 IN THE FOLLOWING SYMMETRIC SPACES:

$$\mathcal{M}_5(J^{\mathbb{R}}_{(1,N)_0}) \subset SL(N+1,\mathbb{R})/SO(N,1)$$

$$\mathcal{M}_5(J^{\mathbb{C}}_{(1,N)_0}) \subset SL(N+1,\mathbb{C})/SU(N,1)$$

$$\mathcal{M}_5(J^{\mathbb{H}}_{(1,N)_0}) \subset SU^*(2N+2)/USp(2N,2)$$

BY DIMENSIONAL REDUCTION TO $d=4$ AND to $d=3$ SPACE-TIME DIMENSIONS ONE OBTAINS COMPLEX AND QUATERNIONIC MANIFOLDS. THEIR ISOMETRY GROUPS ARE IN GENERAL OF THE FORM:

$$\mathcal{M}_4 \quad : \quad Aut(J^{\mathbb{A}}_{(1,N)_0}) \times SO(1,1) \circledS T_{n_V}$$

T_D = TRANSLATION GROUP

$$\mathcal{M}_3 \quad : \quad Aut(J^{\mathbb{A}}_{(1,N)_0}) \times SO(1,1)^2 \circledS \mathcal{H}_{2n_V+3}$$

$\mathcal{H}$ = HEISENBERG GROUP

T_{n_V} = TRANSLATION GROUP

$\mathcal{H}_{2n_V+3}$ = HEISENBERG GROUP

DIMENSIONAL REDUCTION OF 5d, N=2 MESGT

5d, N=2 MESGT WITH n VECTOR MULTIPLETS $\longleftrightarrow$ VERY SPECIAL REAL GEOMETRY $\mathcal{M}_5$ = HYPERSURFACE IN A DOMAIN OF POSITIVITY DEFINED BY C_{IJK}.

$\Downarrow$

4d, N=2 MESGT WITH $(n+1)$ VECTOR MULTIPLETS $\longleftrightarrow$ VERY SPECIAL COMPLEX GEOMETRY $\mathcal{M}_4$ = "UPPER HALF-SPACE" OF THE DOMAIN OF POSITIVITY

$\Downarrow$

3d, N=4 SUGRA COUPLED TO $(n+2)$ N=4 MATTER MULTIPLETS $\longleftrightarrow$ VERY SPECIAL QUATERNIONIC GEOMETRY

GENERIC JORDAN FAMILY OF N=2 MESGT'S

$$J = \mathbb{R} \oplus \Gamma(Q)$$

$$\mathcal{M}_5 = \frac{SO(n-1,1) \times SO(1,1)}{SO(n-1)}$$

$$\mathcal{M}_4 = \frac{SO(n,2)}{SO(n) \times SO(2)} \times \frac{SU(1,1)}{U(1)}$$

$$\mathcal{M}_3 = \frac{SO(n+2,4)}{SO(n+2) \times SO(4)}$$

PURE N=2, d=5 SUGRA UNDER DIMENSIONAL REDUCTION YIELDS N=2, d=4 SUGRA COUPLED TO ONE VECTOR MULTIPLET WITH

$$\mathcal{M}_4 = \frac{SU(1,1)_G}{U(1)} \Rightarrow \left(F^I_{\mu\nu} \oplus \tilde{F}^I_{\mu\nu} \right) \sim s = \tfrac{3}{2} \text{ of } SU(1,1)_G$$

FURTHER REDUCTION TO d=3 YIELDS

$$\mathcal{M}_3 = G_{2(2)} / SU(2) \times SU(2)$$

SYMMETRY GROUPS OF N=2 MESGT's DEFINED BY SIMPLE (FR) JORDAN ALGEBRAS OF DEGREE 3 :

U-Duality	$J_3^{\mathbb{R}}$	$J_3^{\mathbb{C}}$	$J_3^{\mathbb{H}}$	$J_3^{\mathbb{O}}$
K_5	$SO(3)$	$SU(3)$	$USp(6)$	F_4
U_5	$SL(3,\mathbb{R})$	$SL(3,\mathbb{C})$	$SU^*(6)$	$E_{6(-26)}$
U_4	$Sp(6,\mathbb{R})$	$SU(3,3)$	$SO^*(12)$	$E_{7(-25)}$
U_3	$F_{4(4)}$	$E_{6(2)}$	$E_{7(-5)}$	$E_{8(-24)}$

THE "MAGIC SQUARE" OF FREUDENTHAL, ROZENFELD AND TITS $\Rightarrow$ Magical Supergravity theories

	EXCEPTIONAL SUGRA $J_3^{\mathbb{O}}$	MAXIMAL SUGRA	COMMON SECTOR $= J_3^{\mathbb{H}}$ THEORY
$\mathcal{M}_5 =$	$E_{6(-26)}/F_4$	$E_{6(6)}/USp(8)$	$SU^*(6)/USp(6)$
$\mathcal{M}_4 =$	$E_{7(-25)}/E_6 \times U(1)$	$E_{7(7)}/SU(8)$	$SO^*(12)/U(6)$
$\mathcal{M}_3 =$	$E_{8(-24)}/E_7 \times SU(2)$	$E_{8(8)}/SO(16)$	$E_{7(-5)}/SO(12)\times SU(2)$

DISCRETE SUBGROUPS $E_{7(7)}(\mathbb{Z})$ and $E_{6(6)}(\mathbb{Z})$

$\Rightarrow$ SYMMETRY GROUPS OF NON-PERTURBATIVE SPECTRA OF M-THEORY TOROIDALLY COMPACTIFIED

T-DUALITY $\times$ S-DUALITY $\subset$ U-DUALITY

$SO(6,6) \times SL(2,\mathbb{R}) \subset E_{7(7)}$ Hull & Townsend

$SO(5,5) \times SO(1,1) \subset E_{6(6)}$

ANALOGOUS DECOMPOSITION FOR THE EXCEPTIONAL SUGRA

$SO(10,2) \times SL(2,\mathbb{R}) \subset E_{7(-25)}$

$SO(9,1) \times SO(1,1) \subset E_{6(-26)}$

M/STRING THEORETIC ORIGINS OF $N=2$ MESGT'S DEFINED BY JORDAN ALGEBRAS, IN PARTICULAR THE MAGICAL THEORIES ?

DUAL PAIRS OF TYPE II STRING COMPACTIFICATION AS CONSTRUCTED BY SEN + VAFA (1995) USING METHODS DEVELOPED BY VAFA + WITTEN.

ORBIFOLDING $T^4 \times S^1 \times S^1 / \Gamma$:

→ $N=2$, $d=4$ THEORY WITH 15 MASSLESS VECTOR MULTIPLETS AND NO HYPERS CAN BE SHOWN TO BE THE MESGT DEFINED BY $J_3^{\mathbb{H}}$ WITH $\mathcal{M}_4 = \dfrac{SO^*(12)}{U(6)}$

THIS THEORY IS SELF-DUAL WITH THE DILATON BELONGING TO A VECTOR MULTIPLET.

→ GENERIC JORDAN THEORY WITH 7 VECTOR MUL MULTIPLETS

$$\mathcal{M}_4 = \frac{SO(6,2)\times SU(1,1)}{SO(6)\times U(1)\times U(1)}$$

NON-SELF-DUAL

→ STU MODEL WITH 3 VECTOR AND 4 HYPER MULTIPLETS

$$\mathcal{M}_4 = \left[\frac{SU(1,1)}{U(1)}\right]^3 \times \frac{SO(4,4)}{SO(4)\times SO(4)}$$

SELF-DUAL

→ $N=6$ SUPERGRAVITY HAS THE SAME SCALAR MANIFOLD AS THE $N=2$ MESGT DEFINED BY $J_3^{\mathbb{H}}$: $\mathcal{M}_4 = SO^*(12)/U(6)$ + IS SELF-DUAL

— M/SUPERSTRING THEORETIC ORIGIN OF THE EXCEPTIONAL SUPERGRAVITY DEFINED BY $J_3^{\mathbb{O}}$ WITH $\mathcal{M}_4 = \dfrac{E_{7(-25)}}{E_6 \times U(1)}$?

FHSV MODEL:

TYPE II ST ON A SELF-MIRROR CY WITH $h^{11} = h^{12} = 11$ DUAL TO HETEROTIC ST ON $K3 \times T^2$

QUANTUM MODULI ≡ CLASSICAL MODULI

$$\mathcal{M}_4 = \mathcal{M}_V \times \mathcal{M}_H$$

$$\mathcal{M}_V = \frac{SO(10,2) \times SU(1,1)}{SO(10) \times U(1) \times U(1)} \qquad \mathcal{M}_H = \frac{SO(12,4)}{SO(12) \times SO(4)}$$

→ MAXIMAL GENERIC JORDAN THEORY THAT SITS INSIDE THE EXCEPTIONAL MESGT.

$$SO(10,2) \times SU(1,1) \subset E_{7(-25)} \quad + \quad \not\subset E_{7(7)}$$

THERE EXISTS A UNIQUE ANOMALY-FREE SUGRA IN $d=6$ THAT REDUCES TO THE EXCEPTIONAL $N=2$ MESGT COUPLED TO 28 HYPERMULTIPLETS! MG + SEZGIN

$$\mathcal{M}_4 = \frac{E_{7(-25)}}{E_6 \times U(1)} \times \frac{E_{8(-24)}}{E_7 \times SU(2)} = \mathcal{M}_V \times \mathcal{M}_H$$

$$\mathcal{M}_3 = \frac{E_{8(-24)}}{E_7 \times SU(2)} \times \frac{E_{8(-24)}}{E_7 \times SU(2)}$$

MODULI SPACE OF FHSV MODEL IS THE SUBSPACE

$$\frac{SO(12,4)}{SO(12) \times SO(4)} \times \frac{SO(12,4)}{SO(12) \times SO(4)}$$

THE ABOVE OBSERVATIONS SUGGEST THAT THE EXCEPTIONAL MESGT COUPLED TO 28 HYPERS ON $E_{8(-24)} / E_7 \times SU(2)$ COULD BE OBTAINED FROM M-THEORY OVER AN EXCEPTIONAL CY WITH $h_{1,1} = h_{2,1} = 27$.

ANDREY TODOROV (0806.4010 / math.AG)

" CALABI-YAU MANIFOLDS WHOSE MODULI SPACE ARE LOCALLY SYMMETRIC MANIFOLDS AND NO QUANTUM CORRECTIONS "

MODULI SPACE (FOR CY_3) IS $SU(3,3)/S(U(3)\times U(3))$ $(b_2 = 29)$

$\simeq$ THE MODULI SPACE OF 4d $N=2$ MESGT DEFINED BY $J_3^{\mathbb{C}}$.

IMPORTANT QUESTION: CAN ONE EXTEND TODOROV'S CONSTRUCTION TO OTHER CY_3'S WHOSE MODULI SPACES COINCIDE WITH THE SCALAR MANIFOLDS OF THE OTHER MAGICAL SUPERGRAVITY THEORIES COUPLED TO HYPERMULTIPLETS AS WELL AS MESGT'S DEFINED BY LORENTZIAN JORDAN ALGEBRAS !

DOLIVET, JULIA + KOUNNAS: (0712.2867)

DIRECT FERMIONIC CONSTRUCTION OF TYPE II $N=2$ SUPERSTRINGS $\Rightarrow$ 4d MAGICAL SUGRAS WITHOUT HYPERS. (NON-CALABI-YAU)

BIANCHI + FERRARA FURTHER "ARGUE THAT THE EXCEPTIONAL SUPERGRAVITY ADMITS A STRING INTERPRETATION CLOSELY TO THE ENRIQUES (FSHV) MODEL" (0712.297)

ORBITS OF BPS BLACK HOLES IN 5D, N=8 SUPERGRAVITY UNDER THE ACTION OF $E_{6(6)}$

FOR STATIONARY, SPHERICALY SYMMETRIC BPS BLACK HOLES WITH CHARGES q^I $(I=1,\cdots,27)$ ENTROPY IS GIVEN BY THE CUBIC INVARIANT

$$I_3 = C_{IJK}\, q^I q^J q^K \Rightarrow S = \sqrt{I_3}$$

$I_3 \neq 0 \Longrightarrow \frac{1}{8}$ SUSY

Ferrara + Maldacena (97)

$I_3 = 0$ + $\frac{\partial I_3}{\partial q^I} \neq 0 \Rightarrow \frac{1}{4}$ SUSY

$I_3 = 0 = \frac{\partial I_3}{\partial q^I} \Rightarrow \frac{1}{2}$ SUSY

ASSOCIATE WITH A BPS BH SOLUTION WITH CHARGES q^I AN ELEMENT OF SPLIT EXCEPTIONAL JORDAN ALGEBRA $J_3^{O_s}$ (FERRARA + MG 97)

$$J = \sum_{I=1}^{27} e_I\, J^I \qquad \{e_I\} \equiv \text{BASIS OF } J_3^{O_s}$$

$$\mathcal{N}(J) = I_3(q)$$

TIMELIKE ORBIT : $\mathcal{N}(J) > 0$

$$\frac{E_{6(6)}}{F_{4(4)}} = \frac{LG(J_3^{O_s})}{Aut(J_3^{O_s})} \qquad \frac{1}{8} \text{ BPS}$$

LIGHT-LIKE ORBIT :

$$\frac{E_{6(6)}}{O(5,4) \circledS T_{16}} \qquad \frac{1}{4} \text{ BPS}$$

CRITICAL LIGHT-LIKE ORBIT :

$$\frac{E_{6(6)}}{O(5,5) \circledS T_{16}} \qquad \frac{1}{2} \text{ BPS}$$

BPS AND NON-BPS ORBITS FOR EXTREMAL BH's IN $N=2$ MESGT's IN $d=5$ AND $d=4$ WERE GIVEN IN 9708025 (Ferrara + MG)

SOLUTIONS TO ATTRACTOR EQUATIONS FOR NON-BPS ORBITS IN $d=5$ 0606108 (FG)

$$V(\varphi^x, q_I) = q_I \overset{\circ}{a}{}^{IJ} q_J = Z^2 + \frac{3}{2} g^{xy} \partial_x Z \partial_y Z$$ BLACK HOLE POTENTIAL

$Z = q_I h^I$ = CENTRAL CHARGE

J	BPS ORBITS	NON-BPS ORBITS
$\mathbb{R} + \Gamma$	$\frac{SO(n-1,1) \times SO(1,1)}{SO(n-1)}$	$\frac{SO(n-1,1) \times SO(1,1)}{SO(n-2,1)}$
$J_3^{\mathbb{R}}$	$SL(3,\mathbb{R}) / SO(3)$	$SL(3,\mathbb{R}) / SO(2,1)$
$J_3^{\mathbb{C}}$	$SL(3,\mathbb{C}) / SU(3)$	$SL(3,\mathbb{C}) / SU(2,1)$
$J_3^{\mathbb{H}}$	$SU^*(6) / USp(6)$	$SU^*(6) / USp(4,2)$
$J_3^{\mathbb{O}}$	$E_{6(-26)} / F_4$	$E_{6(-26)} / F_{4(-20)}$

CRITICAL POINTS $\Rightarrow$ $\partial_x V \equiv \partial_{\varphi^x} V = 0$

$$\Rightarrow \quad 2Z\partial_x Z - \sqrt{\tfrac{3}{2}}\, T_{xyz}\, \partial^y Z\, \partial^z Z = 0$$

BPS

$\partial_x Z = 0$

$V_{BPS}\big|_{\text{critical}} = Z^2$

$\partial_x \partial_y V\big|_{BPS} = \frac{8}{3} g_{xy} Z^2$

Hessian > 0

NON-BPS

$2Z\partial_x Z = \sqrt{\tfrac{3}{2}}\, T_{xyz}\, \partial^y Z\, \partial^z Z$

$V_{NON\text{-}BPS}\big|_{\text{critical}} = 9 Z^2$

$\partial_x \partial_y V_{NON\text{-}BPS} \geq 0$

positive semi-definite

NOTE: $N=8$ SUGRA IN $d=5$ HAS ONLY ONE ORBIT WITH NON-ZERO ENTROPY

$E_{6(6)} / F_{4(4)}$

DEFINE THE DISTANCE BETWEEN TWO BH SOLUTIONS OF N=8 SUGRA IN $d=5$ (WITH CHARGES $q^I_A + q^I_B$) AS:

$$\mathcal{N}(J_A - J_B) \qquad J_A = q^I_A e_I \quad J_B = q^I_B e_I$$

THE CUBIC DISTANCE FUNCTION IS INVARIANT UNDER THE GENERALIZED "POINCARÉ GROUP"

$$E_{6(6)} \circledS T_{27}$$

THE LIGHT-CONE WITH BASE POINT J_c DEFINED BY THE CONDITION

$$\mathcal{N}(J - J_c) = 0$$

IS INVARIANT UNDER THE CONFORMAL GROUP $E_{7(7)}$!

$E_{7(7)}$ ACTS AS A SPECTRUM GENERATING SYMMETRY IN $d=5$!!

FOR THE N=2 MESGT'S DEFINED BY EUCLIDEAN JORDAN ALGEBRAS OF DEGREE 3 THE ENTROPY OF BPS BH SOLUTIONS ARE ALSO GIVEN BY THE CUBIC NORM OF THE JA

$$S^2 = \mathcal{N}(J) \qquad J = e_I q^I$$

$\frac{1}{2}$ BPS BLACK HOLES OF THE EXCEPTIONAL SUGRA

$$J = e_I q^I \in J_3^{\mathbb{O}}$$

FOR $\mathcal{N}(J) > 0$ TWO POSSIBLE ORBITS

$$E_{6(-26)}/F_4 \qquad\qquad E_{6(-26)}/F_{4(-20)}$$

$$\lambda_1, \lambda_2, \lambda_3 > 0 \qquad\qquad \lambda_1 > 0 \; ; \; \lambda_2, \lambda_3 < 0$$

LIGHT-CONE WITH BASE POINT J_c DEFINED BY THE CONDITION $\mathcal{N}(J - J_c) = 0$ IS INVARIANT UNDER THE CONFORMAL GROUP $E_{7(-25)}$.

DIMENSIONAL REDUCTION TO $d=4$ MESGT's

$d=5$ $\quad A^I_\mu \Longleftrightarrow e^I \in J$.

$d=4$ $\quad F^I_{\mu\nu} \Longleftrightarrow e^I \quad , \quad \tilde{F}^{\mu\nu}_I \Longleftrightarrow e_I \in J$

$\check{A}^0_\mu$ from $d=5$ graviton $\quad F^{\check{0}}_{\mu\nu} \Longleftrightarrow \mathbb{R} \quad , \quad \tilde{F}^{\mu\nu}_{\check{0}} \Longleftrightarrow \mathbb{R}$

$$\begin{pmatrix} F^{\check{0}}_{\mu\nu} & F^I_{\mu\nu} \\ \tilde{F}^{\mu\nu}_I & \tilde{F}^{\mu\nu}_{\check{0}} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \alpha & J \\ \tilde{J} & \beta \end{pmatrix} \in \mathcal{F}(J)$$

FREUDENTHAL TRIPLE SYSTEM DEFINED BY J

$\alpha, \beta \in \mathbb{R}$,

$X, Y, Z \in \mathcal{F}(J) \Rightarrow (X, Y, Z) \in \mathcal{F}(J)$

FREUDENTHAL TRIPLE PRODUCT

$\mathcal{F}(J)$ ADMITS A SYMPLECTIC INVARIANT FORM

$$\langle X, Y \rangle = - \langle Y, X \rangle \in \mathbb{R}$$

AND A QUARTIC INVARIANT (SYMMETRIC)

$$I(X) \equiv \langle (X, X, X), X \rangle$$

FOR THE $N=2$ MESGT's DEFINED BY JORDAN ALGEBRAS :

$$\mathrm{Aut}\,\mathcal{F}(J) = \text{U-DUALITY GROUP IN } d=4$$

$U_4 = G_4$ = ISOMETRY GROUP OF THE SCALAR MANIFOLD $\mathcal{M}_4$

$J = J_3^{\mathbb{O}} \qquad \mathrm{Aut}\,\mathcal{F}(J_3^{\mathbb{O}}) = E_{7(-25)}$

$$\mathcal{M}_4 = E_{7(-25)} / E_6 \times U(1)$$

ORBITS OF EXTREMAL BLACK HOLE SOLUTIONS IN $d=4$

$N=2$ MESGT + $N=8$ THEORY (MG + Ferrara 1997)

$$(F^0_{\mu\nu}, F^I_{\mu\nu}) \oplus (\tilde{F}_0^{\mu\nu}, \tilde{F}_I^{\mu\nu}) \Leftrightarrow$$ vector field strengths + their duals

$I = 1, \ldots, 27$

$$\begin{pmatrix} F^0 & F^I \\ \tilde{F}_J & \tilde{F}_0 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} p^0 & p^I \\ q_J & q_0 \end{pmatrix}$$

p = magnetic charges

q = electric charges

ASSOCIATE WITH A BLACK HOLE SOLUTION WITH ELECTRIC + MAGNETIC CHARGES AN ELEMENT OF $\mathcal{F}(J)$:

$$Q = \begin{pmatrix} p^0 & p^I e_I \\ q_I e^I & q_0 \end{pmatrix} \in \mathcal{F}(J)$$

THE ENTROPY OF THE BLACK HOLE IS GIVEN BY THE QUARTIC INVARIANT OF $\mathcal{F}(J)$

$$S^2 = I_4(Q) = I_4(p^0, p^I, q_0, q_I)$$

FOR $N=8$ SUPERGRAVITY, $J = J_3^{\mathbb{O}_s}$:

TIMELIKE ORBIT : $I_4 > 0$ $\quad E_{7(7)} / E_{6(2)}$

LIGHTLIKE ORBIT: $I_4 = 0$ $\quad E_{7(7)} / F_{4(4)} \circledS T_{26}$

CRITICAL LIGHT-LIKE ORBIT : $I_4 = 0$, $\partial I_4 = 0$ $\quad E_{7(7)} / O(6,5) \circledS \mathcal{H}^{33}$

DOUBLY CRITICAL LIGHT-LIKE ORBIT: $I_4 = 0$, $\partial I_4 = \partial\partial I_4 \big|_{Adj} = 0$ $\quad E_{7(7)} / E_{6(6)} \circledS T_{27}$

$\mathcal{H}^{2n+1}$ = Heisenberg Algebra of dim. $2n+1$

$N=2$, $d=4$ MESGT's

EXTREMAL BLACK ATTRACTOR EQUATIONS

$\Rightarrow$ CRITICALITY CONDITIONS FOR THE BLACK HOLE POTENTIAL V_{BH}

$$V_{BH} \equiv |Z|^2 + G^{I\bar{J}} D_I Z \bar{D}_{\bar{J}} \bar{Z}$$

G = Kähler metric of $\mathcal{M}_4$

Z = covariantly holomorphic central charge function

$$Z = e^{\frac{1}{2}K(z,\bar{z})} \left[q_\Lambda X^\Lambda(z) - p^\Lambda F_\Lambda(z) \right]$$

For symmetric Kähler manifolds $\mathcal{M}_4$

$$D_i C_{jkl} = 0$$

Using the n_V-bein e^i_I one can switch to flat indices $I, J, \ldots$, $\bar{I}, \bar{J}, \ldots$

$$\partial_I V_{BH} = 0 \iff 2\bar{Z} D_I Z + i\tilde{C}_{IJK} \delta^{J\bar{J}} \delta^{K\bar{K}} \bar{D}_{\bar{J}} \bar{Z} \bar{D}_{\bar{K}} \bar{Z} = 0$$

$\tilde{C}_{IJK}$ = $\widetilde{Str}(J)$ invariant tensor

$$X^\Lambda = \begin{pmatrix} X^0 \\ X^I \end{pmatrix} = \begin{pmatrix} 1 \\ z^I \end{pmatrix} \qquad z^I = A^I + i e^\sigma h^I$$

A^I = 4d scalar descending from 5d vectors

(A^0_μ, σ) = (vector + scalar) descending from 5d graviton

$$F_\Lambda = \partial_\Lambda F \qquad F = -\frac{\sqrt{2}}{3} \frac{C_{IJK} X^I X^J X^K}{X^0} = \text{prepotential}$$

$$g_{I\bar{J}} \equiv \partial_I \partial_{\bar{J}} K = \frac{3}{2} e^{-2\sigma} \overset{\circ}{a}_{IJ}$$

Kähler potential $K = -\ln\left[i \bar{X}^\Lambda F_\Lambda - i X^\Lambda \bar{F}_\Lambda \right]$

$$K = -\ln\left\{ i \frac{\sqrt{2}}{3} C_{IJK} (z^I - \bar{z}^I)(z^J - \bar{z}^J)(z^K - \bar{z}^K) \right\}$$

ORBITS OF EXTREMAL BH'S IN 4d, N=2 MESGT WITH $I_4 \neq 0$ (FG 97)

$\mathcal{F}(J)$	$\frac{1}{2}$ BPS ($I_4 > 0$)	NON-BPS ($I_4 < 0$)	NON-BPS ($I_4 > 0$)
$\Gamma_{+}\mathbb{R}$	$\frac{SO(n+2,2)\times SU(1,1)}{SO(n+2)\times SO(2)}$	$\frac{SO(n+2,2)\times SU(1,1)}{SO(n+1,1)\times SO(1,1)}$	$\frac{SO(n+2,2)\times SU(1,1)}{SO(n,2)\times SO(2)}$
$J_3^{\mathbb{O}}$	$\frac{E_{7(-25)}}{E_6}$	$\frac{E_{7(-25)}}{E_{6(-26)}}$	$\frac{E_{7(-25)}}{E_{6(-14)}}$
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{SU(6)}$	$\frac{SO^*(12)}{SU^*(6)}$	$\frac{SO^*(12)}{SU(4,2)}$
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{SU(3)\times SU(3)}$	$\frac{SU(3,3)}{SL(3,\mathbb{C})}$	$\frac{SU(3,3)}{SU(2,1)\times SU(2,1)}$
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{SU(3)}$	$\frac{Sp(6,\mathbb{R})}{SL(3,\mathbb{R})}$	$\frac{Sp(6,\mathbb{R})}{SU(2,1)}$
	$Z \neq 0$, Hessian > 0	$Z \neq 0$, Hessian $\geqslant 0$	$Z = 0$, Hessian $\geqslant 0$

$$V_{BH} = |Z|^2 + G^{I\bar{J}} D_I Z \, \overline{D_J Z}$$

Bellucci, Ferrara, M.G + Marrani (2006)

Z = central charge

$$Z = e^{K(z,\bar{z})/2}\left(X^{\Lambda}(z)\, q_\Lambda - F_\Lambda(z)\, p^\Lambda\right) = L^\Lambda q_\Lambda - M_\Lambda p^\Lambda$$

$K(z,\bar{z})$ = Kähler potential , D_I = Kähler Covariant Derivative

CRITICAL POINTS :

$$\partial_I V_{BH} = 0 \Rightarrow 2\bar{Z} D_I Z + i C_{IJK} G^{J\bar{J}} G^{K\bar{K}} \bar{D}_{\bar{J}}\bar{Z}\, \bar{D}_{\bar{K}}\bar{Z} = 0$$

BPS $\Rightarrow$ $D_I Z = 0$ NON-BPS : $D_I Z \neq 0$

FOUR DIMENSIONAL U-DUALITY GROUP $\mathrm{Conf}(J)$ LEAVES THE LIGHT-CONE DEFINED BY THE CUBIC NORM $N(J)$ INVARIANT AND ACTS AS SPECTRUM GENERATING SYMMETRY GROUP OF FIVE DIMENSIONAL EXTREMAL BLACK HOLES!

$$\mathrm{Conf}(J) = \underset{\text{SPECIAL CONFORMAL TRANSFS.}}{K_J} \oplus \underbrace{(LG \times \mathcal{D})}_{\text{LORENTZ GROUP + DILATIONS}} \oplus \underset{\text{TRANSLATIONS}}{T_J}$$

COULD THE 3-DIMENSIONAL U-DUALITY GROUP U_3 SIMILARLY ACT AS SPECTRUM GENERATING GROUP OF 4-DIMENSIONAL EXTREMAL BLACK HOLES?

MG, KOEPSELL + NICOLAI

PROBLEM: E_8, F_4 AND G_2 DO NOT ADMIT A 3-GRADING WITH RESPECT TO ANY SUBGROUP OF MAXIMAL RANK $\Rightarrow$ THEY CAN NOT BE REALIZED AS GENERALIZED CONFORMAL GROUPS!

EXCEPT FOR A_1, ALL SIMPLE LIE ALGEBRAS ADMIT A 5-GRADING W.R.T A SUBALGEBRA g^0 OF MAXIMAL RANK SUCH THAT $\dim g^{\mp 2} = 1$

$$g = g^{-2} \oplus g^{-1} \oplus g^{0} \oplus \underset{\mathcal{F}(J)}{\underset{\Updownarrow}{g^{+1}}} \oplus \underset{\mathbb{R}}{\underset{\Updownarrow}{g^{+2}}}$$

SUCH LIE ALGEBRAS CAN BE REALIZED IN TERMS OF AN UNDERLYING FTS.

$$g^0 = \mathrm{Aut}(\mathcal{F}(J)) \oplus SO(1,1)$$

$$\mathrm{Aut}(\mathcal{F}(J)) \equiv U_4 = \text{4d U-DUALITY GROUP} \approx \mathrm{Conf}(J)$$

ALL SIMPLE LIE ALGEBRAS INCLUDING G_2, F_4, E_8 ADMIT A 5-GRADED DECOMPOSITION w.r.t. A SUBALGEBRA g^0 OF MAXIMAL RANK

$$g = g^{-2} \oplus g^{-1} \oplus g^{0} \oplus g^{+1} \oplus g^{+2} \quad \ni \quad \boxed{\dim g^{\pm 2} = 1}$$

THEY CAN BE REALIZED OVER FREUDENTAL-KANTOR TRIPLE SYSTEMS $\mathcal{F} \Longleftrightarrow$ SUBSPACE g^{+1} of g.

FREUDENTHAL'S CONSTRUCTION OF F_4, E_6, E_7, E_8

$$\mathcal{F}(J_3^{A}) = \begin{pmatrix} \alpha & J_3^{A} \\ \tilde{J}_3^{A} & \beta \end{pmatrix}$$

$A = \mathbb{R} \longleftrightarrow F_4$

$A = \mathbb{C} \longleftrightarrow E_6$

$A = \mathbb{H} \longleftrightarrow E_7$

$A = \mathbb{O} \longleftrightarrow E_8$

FREUDENTHAL TRIPLE PRODUCT:

$(X, Y, Z) \in \mathcal{F}(J)$

$\forall \quad X, Y, Z \in \mathcal{F}(J)$

Automorphism Groups of $\mathcal{F}(J_3^{A})$

$\mathrm{Aut}\, \mathcal{F}(J_3^{O_s}) \approx E_{7(7)}$

$\mathrm{Aut}\, \mathcal{F}(J_3^{O}) \approx E_{7(-25)}$

$\mathrm{Aut}\, \mathcal{F}(J_3^{H}) \approx SO^*(12)$

$\mathrm{Aut}\, \mathcal{F}(J_3^{C}) \approx SU(3,3)$

$\mathrm{Aut}\, \mathcal{F}(J_3^{R}) \approx Sp(6,\mathbb{R})$

$\mathrm{Aut}\, \mathcal{F}(\mathbb{R} + \Gamma(d)) \approx SO(d,2) \times SO(2,1)$

$\mathcal{F}$ ADMITS A SYMPLECTIC INVARIANT FORM

$$\langle X, Y \rangle = - \langle Y, X \rangle \qquad X, Y \in \mathcal{F}$$

SUCH THAT THE QUARTIC INVARIANT I_4 OF $\mathcal{F}$ IS:

$$I_4(X) \equiv \langle (X, X, X), X \rangle$$

NOTE THAT $\mathrm{Aut}(\mathcal{F}(J_3)) \cong$ Conformal Group of J_3

SUBALGEBRA $g^0 = \mathrm{Aut}(\mathcal{F}(J)) \times SO(1,1)$

QUASICONFORMAL REALIZATION OF $E_{8(8)}$

M.G, KOEPSELL + NICOLAI (2000)

$$E_{8(8)} = \underbrace{\overset{x}{\updownarrow}{1} \oplus \overset{\bar{X}}{\updownarrow}{56}} \oplus \left(E_{7(7)} + D\right) \oplus \widetilde{56} \oplus \tilde{1}$$

$$E_{8(8)} = K \oplus U_A \oplus S_{AB} \oplus \tilde{U}_A \oplus \tilde{K} \, , \qquad A, B, \ldots \in \mathcal{F}(J)$$

CONSIDER THE ACTION OF $E_{8(8)}$ ON A 57 DIMENSIONAL SPACE WITH COORDINATES:

$$\mathcal{X} = (\bar{X}, x) \qquad \bar{X} \in \mathcal{F}(J_3^{O_s}) \qquad x = \text{singlet}$$

$E_{8(8)}$ ACTION ON $\mathcal{X}$:

$$K(\bar{X}) = 0 \qquad K(x) = 2a$$

$$U_A(\bar{X}) = A \qquad U_A(x) = \langle A, \bar{X} \rangle$$

$$S_{AB}(\bar{X}) = (A, B, \bar{X}) \qquad S_{AB}(x) = 2\langle A, B \rangle x$$

$$\tilde{U}_A(\bar{X}) = \frac{1}{2}(\bar{X}, A, \bar{X}) - A x$$

$$\tilde{U}_A(x) = -\frac{1}{6}\langle (\bar{X}, \bar{X}, \bar{X}), A \rangle + \langle \bar{X}, A \rangle x$$

$$\tilde{K}(\bar{X}) = -\frac{1}{6}(\bar{X}, \bar{X}, \bar{X}) + \bar{X} x$$

$$\tilde{K}(x) = \frac{1}{6}\langle (\bar{X}, \bar{X}, \bar{X}), \bar{X} \rangle + 2x^2$$

(A, B, C) = FREUDENTHAL TRIPLE PRODUCT

$(U_A \oplus K)$ FORM A 57 DIMENSIONAL HEISENBERG ALGEBRA OF $E_{8(8)}$

$$[U_A, U_B] = \langle A, B \rangle K$$

SIMILARLY, $(\tilde{U}_A \oplus \tilde{K})$ FORM AN HEISENBERG ALGEBRA.

$\mathbb{X} = (X, x) \quad , \quad \mathbb{Y} = (Y, y) \qquad X, Y \in \mathcal{F}(J_3^{\mathbb{O}_s})$

DEFINE "SYMPLECTIC DIFFERENCE" $\ominus$ OF 57-VECTORS $\mathbb{X}$ + $\mathbb{Y}$ AS

$$\mathbb{X} \ominus \mathbb{Y} \equiv (X - Y \, , \, x - y + \langle X, Y \rangle) = -\mathbb{Y} \ominus \mathbb{X}$$

AND THE QUARTIC NORM $\mathcal{N}_4$ OF A 57-VECTOR AS

$$\mathcal{N}_4(\mathbb{X}) \equiv 4 I_4(X) - x^2$$

$\mathcal{N}_4$ IS MANIFESTLY INVARIANT UNDER $E_{7(7)}$.

LIGHT-CONE WITH BASE POINT $\mathbb{Y}_B$ DEFINED BY THE SET OF 57-VECTORS $\mathbb{X}$

$$\mathcal{N}_4(\mathbb{X} \ominus \mathbb{Y}_B) = 0$$

IS INVARIANT UNDER THE ABOVE ACTION OF $E_{8(8)}$.

$\Rightarrow$ FIRST KNOWN GEOMETRIC REALIZATION OF $E_{8(8)}$ AS THE INVARIANCE GROUP OF A LIGHT-CONE IN 57 DIMENSIONS DEFINED BY A QUARTIC DISTANCE FUNCTION.

$\Rightarrow$ THIS REALIZATION EXTENDS TO COMPLEX E_8 AND HENCE TO ALL ITS REAL FORMS!

QUASICONFORMAL REALIZATIONS OF OTHER SPLIT EXCEPTIONAL GROUPS

$$E_{7(7)} = \bar{1} \oplus \overline{32} \oplus (SO(6,6) + D) \oplus \underbrace{32 \oplus 1}$$

$$E_{6(6)} = \bar{1} \oplus \overline{20} \oplus (SL(6,\mathbb{R}) + D) \oplus \underbrace{20 \oplus 1}$$

$$F_{4(4)} = \bar{1} \oplus \overline{14} \oplus (Sp(6,\mathbb{R}) + D) \oplus \underbrace{14 \oplus 1}$$

$$G_{2(2)} = \bar{1} \oplus \bar{4} \oplus (SL(2,\mathbb{R}) + D) \oplus \underbrace{4 \oplus 1}$$

IDENTIFY THE CHARGE-ENTROPY SPACE OF EXTREMAL BLACK HOLES OF N=8 SUGRA WITH THE 57 DIMENSIONAL SPACE ON WHICH $E_{8(8)}$ ACTS AS THE QUASICONFORMAL GROUP

$$\mathbb{Q} = (Q_A, s) \qquad Q_A = (q_0, q_I, p^0, p^I) \qquad I = 1, \dots, 27$$

s = entropy $\qquad$ Q_A = charges

$$N_4(\mathbb{Q}) = I_4(q,p) - s^2$$

LIGHT-CONE: $\quad N_4(\mathbb{Q}) = 0 \Rightarrow s^2 = I_4(p,q)$

DEFINE THE DISTANCE BETWEEN TWO BLACK HOLE SOLUTIONS IN CHARGE-ENTROPY SPACE AS

$$d(\mathbb{Q}, \tilde{\mathbb{Q}}) = N_4(\mathbb{Q} \ominus \tilde{\mathbb{Q}})$$

LIGHT-LIKE SEPARATIONS ARE LEFT INVARIANT UNDER THE QUASICONFORMAL ACTION OF $E_{8(8)}$.

PROPOSAL: $E_{8(8)}$ ACTS AS SPECTRUM GENERATING SYMMETRY GROUP OF EXTREMAL BLACK HOLES OF N = 8 SUGRA IN $d=4$ $\qquad$ GKN

NOTE THAT $E_{8(8)} \approx U_3$ = U-DUALITY GROUP OF N=8 SUGRA IN $d=3$

RECALL THE PROPOSAL THAT $U_4 = E_{7(7)}$ ACTS AS THE SPECTRUM GENERATING CONFORMAL GROUP IN 5D, N=8 SUGRA.

SIMILARLY, THE U-DUALITY GROUPS U_3 OF N=2 MESGT'S DEFINED BY JORDAN ALGEBRAS J OF DEGREE 3 SHOULD ACT AS SPECTRUM GENERATING QUASICONFORMAL GROUPS OF THE CORRESPONDING N=2 MESGT'S IN $d=4$.

THE MINIMAL UNITARY REPRESENTATION OF $E_{8(8)}$

OBTAINED BY QUANTIZATION OF THE QUASICONFORMAL

REALIZATION : GKN 2000

$$E_{8(8)} = E \oplus \begin{pmatrix} E_{ij} \\ E^{ij} \end{pmatrix} \oplus \left(E_{7(7)} + D\right) \oplus \begin{pmatrix} F_{ij} \\ F^{ij} \end{pmatrix} \oplus F$$

$$248 = 1 \oplus (28+\widetilde{28}) \oplus (133+1) \oplus (28+\widetilde{28}) + 1$$

28 COORDINATES $X^{ij} = -X^{ji}$ $(i,j = 1,\dots 8)$

28 MOMENTA $P_{ij} = -P_{ji}$

$$[X^{ij}, P_{kl}] = i\, \delta^{ij}_{kl}$$ SL(8,R) Basis

$$\begin{pmatrix} E^{ij} \\ E_{ij} \end{pmatrix} = \begin{pmatrix} x\, X^{ij} \\ x\, P_{ij} \end{pmatrix} \in g^{-1} \quad \Rightarrow \quad E = \tfrac{1}{2} x^2 \in g^{-2}$$

$$\left(G^i_j \oplus G^{ijkl}\right) \in E_{7(7)} \in g^0$$

$$G^i_j = 2X^{ik} P_{kj} + \frac{1}{4} X^{kl} P_{kl}\, \delta^i_j \qquad SU(8)$$

$$G^{ijkl} = -\frac{1}{2} X^{[ij} X^{kl]} + \frac{1}{48} \epsilon^{ijklmnpq} P_{mn} P_{pq} \qquad \frac{E_{7(7)}}{SU(8)}$$

MOMENTUM p CONJUGATE TO x $[x, p] = i$

$$\begin{pmatrix} F^{ij} \\ F_{ij} \end{pmatrix} \in g^{+1} \qquad F \in g^{+2}$$

$$F^{ij} = -p\, X^{ij} + \frac{2i}{x}\left[X^{ij}, I_4(X,P)\right]$$

$$F_{ij} = -p\, P_{ij} + \frac{2i}{x}\left[P_{ij}, I_4(X,P)\right]$$

$$F = \frac{p^2}{2} + \frac{2 I_4}{x^2}, \qquad D = \frac{1}{2}(xp + px)$$

UNITARY REPRESENTATION ON THE HILBERT SPACE OF SQUARE INTEGRABLE FUNCTIONS IN 29 VARIABLES

$(X_{ij}, x) \Rightarrow$ MINIMAL UIR OF $E_{8(8)}$.

SL(2,ℝ) IN THE MINIMAL UIR OF $E_{8(8)}$

$$E = \frac{1}{2}x^2 \,, \quad D = \frac{1}{2}(xp+px) \,, \quad F = \frac{p^2}{2} + \frac{2I_4}{x^2}$$

CONFORMAL QUANTUM MECHANICS

$I_4(X,P) \Longleftrightarrow$ COUPLING CONSTANT g

SL(2,ℝ) AND $E_{7(7)}$ FORM A DUAL PAIR IN $E_{8(8)}$

QUADRATIC CASIMIRS :

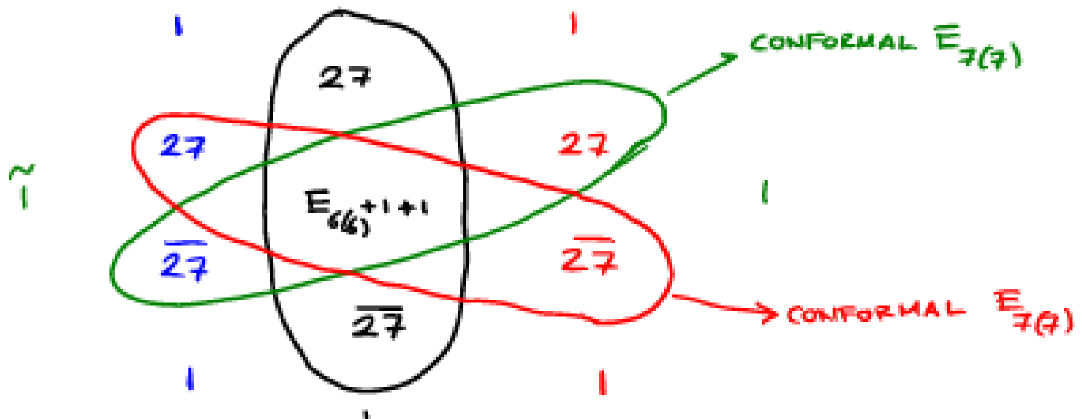
$$\mathcal{C}_2(SL(2,\mathbb{R})) = I_4 - \frac{3}{16}$$

$$\mathcal{C}_2(E_{7(7)}) = 3I_4 - \frac{969}{16}$$

$$\mathcal{C}_2(E_{8(8)}) = -120$$

MINIMAL UIR OF $E_{8(8)}$ DECOMPOSES INTO INFINITELY MANY IRREPS OF $E_{7(7)} \times SL(2,\mathbb{R})$.

$$E_{8(8)} = \tilde{1} \oplus \widetilde{56} \oplus (E_{7(7)} + D) \oplus 56 \oplus 1$$



$E_{7(7)}$ ACTS LINEARLY ON $g^{+1} = 56$

$E_{7(7)}$ CONFORMAL OVER 27 COORDINATES

$E_{7(7)}$ CONFORMAL OVER $\widetilde{27}$ MOMENTA

NOTE CONFORMAL REALIZATION VERSUS MINIMAL UIR :

$$E_{7(7)} = \tilde{1} \oplus \widetilde{32} \oplus (SO(6,6)+D) \oplus 32 \oplus 1$$

$32 = 16 + \widetilde{16} \Rightarrow$ MINREP OVER $L^2(16+1)$.

QUASICONFORMAL REALIZATION OF THE U-DUALITY GROUPS U_3 OF $N=4$ MESGT'S IN $d=3$ DEFINED BY JORDAN ALGEBRAS OF DEGREE 3

M.G. + PAVLYK

SCALAR MANIFOLDS ARE QUATERNIONIC SYMMETRIC

$$\mathcal{M}_3(\mathbb{R}+\Gamma(n)) = \frac{SO(n+2,4)}{SO(n+2)\times SO(4)}$$

$$\mathcal{M}_3(J_3^{\mathbb{R}}) = \frac{F_{4(4)}}{Usp(6)\times SU(2)} \qquad \mathcal{M}_3(J_3^{\mathbb{C}}) = \frac{E_{6(2)}}{SU(6)\times SU(2)}$$

$$\mathcal{M}_3(J_3^{\mathbb{H}}) = \frac{E_{7(-5)}}{SO(12)\times SU(2)} \qquad \mathcal{M}_3(J_3^{\mathbb{O}}) = \frac{E_{8(-24)}}{E_7\times SU(2)}$$

NOTE $$\mathcal{M}_3(J) = \frac{QConf(J)}{\widetilde{Conf}(J)\times SU(2)}$$

$\widetilde{Conf}(J)$ = COMPACT REAL FORM OF $Conf(J)$

$QConf(J) \Rightarrow$ SPECTRUM GENERATING SYMMETRY OF 4D, $N=2$ MESGT

MINIMAL UIR'S OF $QConf(J)$ OBTAINED BY QUANTIZING THE GEOMETRIC ACTION (M.G + Pavlyk)

$$E_{8(-24)} = 1 \oplus 56 \oplus (E_{7(-25)} + D) \oplus 56 \oplus 1$$

$$E_{8(8)} = 1 \oplus 56 \oplus (E_{7(7)} + D) \oplus 56 \oplus 1$$

$$E_{8(8)} \supset SO(16) \qquad E_{8(-24)} \supset E_7 \times SU(2)$$

$$E_{7(7)} \supset SU(8) \qquad E_{7(-25)} \supset SU(6,2)$$

$$E_{7(7)} \supset SL(8,\mathbb{R}) \qquad E_{7(-25)} \supset SU^*(8)$$

THE METHOD OF OBTAINING THE MINIMAL UNITARY REPRESENTATION BY QUANTIZATION OF QUASICONFORMAL REALIZATION EXTENDS TO ALL NON-COMPACT GROUPS AND SUPERGROUPS (M.G + Pavlyk)

$$QConf = K \oplus U_\Lambda \oplus (S_{(\Lambda\Sigma)} + \Delta) \oplus \tilde{U}_\Lambda \oplus \tilde{K}$$

$$K = \frac{y^2}{2} \qquad \Delta = \frac{1}{2}(yp+py) \qquad \tilde{K} = \frac{p^2}{2} + \frac{2I_4}{y^2}$$

$\{K, \Delta, \tilde{K}\}$ GENERATE $SL(2,\mathbb{R})$ OF CONFORMAL QUANTUM MECHANICS (Fubini et.al)

$$[y,p]=i \qquad I_4 = I_4(X,P) = \text{coupling constant}$$

$$U_\Lambda = \begin{pmatrix} y X_A \\ y P^A \end{pmatrix} \qquad [X_A, P^B] = i\,\delta_A^B$$

$I_4 \Rightarrow C_2(H)$ = quadratic Casimir of H generated by $S_{(AB)}$.

$S_{[\Lambda\Sigma]} \propto \Omega_{\Lambda\Sigma}\,\Delta \qquad \Omega_{\Lambda\Sigma}$ = Symplectic metric of H.

FOR $QConf(J)$: $\quad S_{(\Lambda\Sigma)} \cong Conf(J)$

$$Conf(J) = R_I \oplus (R_I^J + \mathcal{R}) \oplus \tilde{R}^I = Aut\,\mathcal{F}(J)$$

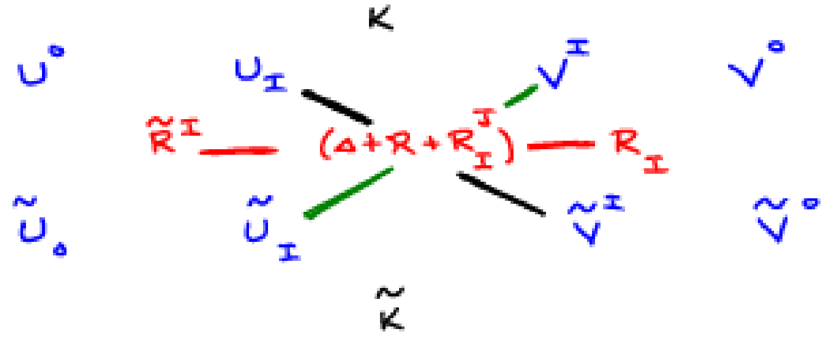
$QConf(J)$ has 7-grading w.r.t $\mathcal{R}$

$$X_A = (q_0, q_I) \qquad P^A = (p^0, p^I) \quad , \quad J = e^I q_I$$

$$I_4(X,P) = p^0 N(q) + q_0 N(p) + (p^0 q_0 - p^I q_I)^2 + q^I_{\#}\, p^{\#}_I$$

I_4 differs from the quadratic Casimir of $Conf(J)$ by a c-number due to normal ordering!

WITH RESPECT TO $(\mathcal{R}, \Delta)$ THE LIE ALGEBRA OF $QCon(J)$ HAS A $(7,5)$-GRADED STRUCTURE



$R_I^I = 0 \rightarrow R_I^J$ = GENERATORS OF LORENTZ GROUP OF $J \cong U_5$

MINIMAL UNITARY REALIZATION OF $QCONF(J)$ OVER THE HILBERT SPACE OF SQUARE INTEGRABLE FUNCTIONS IN $D = \dim(J) + 2$ VARIABLES:

$Conf(J)$ GENERATED BY R_I, $\tilde{R}^J$, $[R_I, \tilde{R}^J]$ IS REALIZED AS BILINEARS OF POSITION (q_I, q_0, y) AND MOMENTUM (p^I, p^0, p) OPERATORS.

$CONF(J)$ GENERATED BY U_I, $\tilde{V}^J$ AND $[U_I, \tilde{V}^J]$ IS REALIZED NON-LINEARLY.

SIMILARLY, $CONF(J)$ GENERATED BY $\tilde{U}_I$, V^J AND $[\tilde{U}_I, V^J]$ IS ALSO REALIZED NONLINEARLY.

$QConf(J_3^{\mathbb{O}}) = E_{8(-24)}$	$QConf(J_3^{\mathbb{O}_s}) = E_{8(8)}$
$Conf(J_3^{\mathbb{O}}) = E_{7(-25)}$	$Conf(J_3^{\mathbb{O}_s}) = E_{7(7)}$
$Lor(J_3^{\mathbb{O}}) = E_{6(-26)}$	$Lor(J_3^{\mathbb{O}_s}) = E_{6(6)}$
$Aut(J_3^{\mathbb{O}}) = F_4$	$Aut(J_3^{\mathbb{O}_s}) = F_{4(4)}$

QUANTUM ATTRACTOR FLOWS AND $U_3 = G_3$ AS SPECTRUM GENERATING SYMMETRY GROUP OF SSS BPS BLACK HOLES OF 4d MESGT's

M.G, Neitzke, Pioline & Waldron

0512296, 0707.0267

Attractor equations for a SSS BPS BH IN $d=4, N=2$ MESGT $\approx$ Geodesic motion of a fiducial particle on the scalar manifold $\mathcal{M}_3^*$ of the 3d sugra obtained by reduction on a time-like circle.

Breitenlohner
Gibbons
Maison (1988)

$d=4$ $\qquad \mathcal{M}_4 = \dfrac{Conf(J)}{\widetilde{Str}(J)}$

$$\begin{tikzcd}[column sep=small] & \mathcal{M}_4 \arrow[dl, "\text{spacelike circle}"'] \arrow[dr, "\text{timelike circle}"] & \\ \mathcal{M}_3 = \dfrac{QConf(J)}{\widetilde{Conf}(J)\times SU(2)} & & \mathcal{M}_3^* = \dfrac{QConf(J)}{Conf(J)\times SU(1,1)} \end{tikzcd}$$

quaternionic Kähler $\qquad\qquad$ para-quaternionic Kähler

(RADIAL) QUANTIZATION OF THE GEODESIC MOTION ON $\mathcal{M}_3^*$

$\Rightarrow$ HILBERT SPACE $\equiv L^2(\mathcal{M}_3^*)$

GENERAL PHASE SPACE $\equiv$ TANGENT SPACE $T(\mathcal{M}_3^*)$

$\Rightarrow$ UNITARY REALIZATION OF $QConf(J)$ OVER $L^2(\mathcal{M}_3^*)$

BPS SUBSPACE OF $T(\mathcal{M}_3^*) \Rightarrow$ TWISTOR SPACE $\mathcal{Z}_3^*$ OF $\mathcal{M}_3^*$

QUASICONFORMAL REALIZATION $\Rightarrow$ ACTION OF $G_3 = U_3$ ON $G_3/\mathcal{P}$

$\mathcal{P}$ = HEISENBERG PARABOLIC SUBGROUP GENERATED BY $(\mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0})$

TWISTOR SPACE $\mathcal{Z}_3 = G_3^{\mathbb{C}}/\mathcal{P}^{\mathbb{C}}$

QUASICONFORMAL REALIZATION $\Rightarrow$ HOLOMORPHIC ACTION OF $QCON(J)$ ON $\mathcal{Z}_3$

$G_3 = U_3 = QCONF(J)$.

QUANTUM HILBERT SPACE OF BPS BLACK HOLES $\Rightarrow$ UNITARY REPRESENTATIONS OVER $\mathcal{Z}$ INDUCED BY QCG ACTION TWISTED BY A UNITARY CHARACTER.

N=2 SUPERSYMMETRIC SIGMA MODELS THAT COUPLE TO N=2 SUPERGRAVITY IN HARMONIC SUPERSPACE

TARGET SPACE IS A QUATERNIONIC KÄHLER MANIFOLD (BAGGER + WITTEN 1983)

HSS FORMULATION (GALPERIN + OGIEVETSKY 1992)

$$S = \int d\zeta^{-4} du \left\{ Q^+_\alpha D^{++} Q^{+\alpha} - q^+_i D^{++} q^{+i} + \mathcal{L}^{+4}(Q^+, q^+, u^-) \right\}$$

$\zeta^M = \{ x^A_\mu, \theta^{a+}, \bar{\theta}^{\dot{a}+} \}, u^\pm_i$ ARE COORDINATES OF ANALYTIC SUPERSPACE: $\quad u^{+i} u^-_i = 1 \qquad i=1,2$

$$D^{++} u^-_i = u^+_i$$

$Q^+_\alpha(\zeta, u)$, $\alpha = 1, \dots, 2n$ hypermultiplets

$q^+_i(\zeta, u)$, $i = 1,2$ sugra hypermultiplet compensators

THE ACTION S INVOLVES A SINGLE DERIVATIVE D^{++} AND HAS THE FORM OF HAMILTONIAN MECHANICS

$$\mathcal{L}^{+4} = \frac{\mathcal{P}^{+4}(Q)}{(q^+ u^-)^2} \quad , \quad \mathcal{P}^{+4} = \frac{1}{12} S_{\alpha\beta\gamma\delta} Q^{+\alpha} Q^{+\beta} Q^{+\gamma} Q^{+\delta}$$

$S_{\alpha\beta\gamma\delta}$ IS SYMMETRIC

ISOMETRIES OF THE TARGET MANIFOLD ARE GENERATED BY KILLING POTENTIALS K^{++}_A WHICH OBEY THE THE CONSERVATION LAW:

$$\partial^{++} K^{++}_A + \{ K^{++}_A, \mathcal{L}^{+4} \} = 0 \qquad \partial^{++} = u^{+i} \frac{\partial}{\partial u^{-i}}$$

& GENERATE THE LIE ALGEBRA OF THE ISOMETRY GROUP

$$\{ K^{++}_A, K^{++}_B \} = f_{AB}{}^{C} K^{++}_C \qquad \{ , \} = \text{P.B.}$$

CONSIDER SYMMETRIC TARGET SPACES

$$G/_{H \times SU(2)} \qquad K^{++}_A \Longleftrightarrow \text{GENERATORS OF } G$$

REMARKABLE MAPPING BETWEEN HSS FORMULATION OF N=2 SIGMA MODEL THAT COUPLES TO SUGRA AND THE MINIMAL UNITARY REALIZATIONS OF THEIR ISOMETRY GROUPS

M.G. 2007

$$\mathcal{M} = G/H\times SU(2)$$

HSS FORMULATION	MINIMAL UNITARY REALIZATION OF G
$x_c = (q^{+i}u^-_i),\ (q^{+i}u^+_i) = p_c$	$x,\ p$
$Q^{+\alpha}\ (\alpha = 1,\dots,2n)$	$\xi^\alpha = (X_A, P^B)\quad A,B = 1,\dots,n$
$\{\ ,\ \}_{P.B}$	$i[\ ,\ ]$
$\mathcal{P}^{+4}(Q)$	$I_4(X,P)$
$J^{++}_a = (t_a)_{\alpha\beta} Q^{+\alpha} Q^{+\beta}$	$J_a = (t_a)_{\alpha\beta}\xi^\alpha\xi^\beta$
$K = x_c^2$	$K = x^2$
$\tilde{K} = p_c^2 - \frac{2\mathcal{P}^{+4}}{x_c^2}$	$\tilde{K} = p^2 - \frac{I_4(X,P)}{y^2}$
$K_\alpha = x_c Q^+_\alpha$	$K_\alpha = x\,\xi_\alpha$
$\tilde{K}_\alpha = [\tilde{K}, K_\alpha]$	$\tilde{K}_\alpha = [\tilde{K}, K_\alpha]$
$\Delta = y_c p_c + p_c y_c$	$\Delta = yp + py$

$$g = K \oplus K_\alpha \oplus (J_a + \Delta) \oplus \tilde{K}_\alpha \oplus \tilde{K}$$

FUNDAMENTAL SPECTRUM OF THE QUANTUM N=2 SIGMA MODEL MUST BELONG TO THE MINIMAL UIR OF ITS ISOMETRY GROUP G.

FULL QUANTUM SPECTRUM MUST BELONG TO THE REPRESENTATIONS FORMED BY TENSORING OF MINIMAL UIR OF G!

UNITARY REPRESENTATIONS INDUCED BY THE GEOMETRIC QUASICONFORMAL ACTION TWISTED BY A UNITARY CHARACTER : (MG, NEITZKE, PAVLYK, PIOLINE (2007))

DETAILED STUDY OF RANK 2 QUATERNIONIC GROUPS $SU(2,1)$ AND $G_{2(2)}$. THESE REPRESENTIONS INCLUDE QUATERNIONIC DISCRETE SERIES REPS OF GROSS & WALLACH.

$\frac{1}{8}$ BPS BLACK HOLES $\Rightarrow$ QUATERNIONIC DISCRETE SERIES

HARMONIC SUPERSPACE $\Rightarrow$ MINIMAL UNITARY REPRESENTATIONS $\oplus$ THEIR PRODUCTS

MINREP IS NOT IN THE QDS. IT IS IN THE SINGULAR CONTINUATION OF QDS.

– PHYSICALLY RELEVANT REPS ARE THE MINIMAL UIR'S AND THOSE OBTAINABLE BY TENSORING OF MINREPS

c.f. $AdS/_{CFT} \overset{?}{\Longrightarrow} QDS/_{MINREP}$

SINGLETONS DO NOT BELONG TO THE HOLOMORPHIC DISCRETE SERIES BUT ARE IN A SINGULAR CONTINUATION OF HERMITIAN SYMMETRIC NONCOMPACT GROUPS.

HERMITIAN SYMMETRIC REAL FORM:	QUATERNIONIC SYMMETRIC REAL FORM:
$Conf(J)$ WITH MAXIMAL COMPACT SUBGROUP $\widetilde{Str}_0(J)\times U(1)$	$QConf(J)$ WITH MAXIMAL COMPACT SUBGRP $\widetilde{Conf}(J)\times SU(2)$
$\Downarrow$	$\Downarrow$
HOLOMORPHIC DISCRETE SERIES	QUATERNIONIC DISCRETE SERIES
$\Downarrow$	$\Downarrow$
SINGLETONS + DOUBLETONS IN THE CONTINUATION OF HDS	MINREP IN THE CONTINUATION OF QDS
$\Downarrow$	$\Downarrow$
TENSORING SINGLETONS + DOUBLETONS YIELD THE ENTIRE HDS !	TENSORING OF MINREP YIELDS THE ENTIRE QDS ?

e.f. $AdS_5\times S^5$ (GM 84), $AdS_4\times S^7$ (GW84), $AdS_7\times S^4$ (GvNW 84)

FUTURE DIRECTIONS + OPEN PROBLEMS

- CAN ONE OBTAIN THE ENTIRE QUATERNIONIC DISCRETE SERIES BY TENSORING THE MINREPS ? (IN PROGRESS)
- UNITARY REPRESENTATIONS OF THE DISCRETE ARITHMETIC SUBGROUPS OF U-DUALITY GROUPS ?
- QUANTIZATION OF THE QUATERNIONIC KÄHLER SIGMA MODELS IN HARMONIC SUPERSPACE ?
- GEOMETRIES OF THE MESGT'S DEFINED BY LORENTZIAN JORDAN ALGEBRAS IN FOUR AND THREE SPACE-TIME DIMENSIONS.

  COMPLEX + QUATERNIONIC "ISOPARAMETRIC" SUBMANIFOLDS THAT GENERALIZE CARTAN'S REMARKABLE HYPERSURFACES.
- EXTENSION OF RESULTS OF GNPP TO ALL NON-COMPACT QUATERNIONIC GROUPS (IN PROGRESS WITH PAVLYK)